Bosons in Disc-Shaped Traps: From 3D to 2D

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Abstract

We present a mathematically rigorous analysis of the ground state of a dilute, interacting Bose gas in a three-dimensional trap that is strongly confining in one direction so that the system becomes effectively two-dimensional. The parameters involved are the particle number, $N \gg 1$, the two-dimensional extension, $\bar{L}$, of the gas cloud in the trap, the thickness, $h \ll \bar{L}$ of the trap, and the scattering length a of the interaction potential. Our analysis starts from the full many-body Hamiltonian with an interaction potential that is assumed to be repulsive, radially symmetric and of short range, but otherwise arbitrary. In particular, hard cores are allowed. Under the premises that the confining energy, $\sim 1/h^2$, is much larger than the internal energy per particle, and $a/h \to 0$, we prove that the system can be treated as a gas of two-dimensional bosons with scattering length $a_{2\mathrm{D}} = h\exp(-(\text{const.})h/a)$. In the parameter region where $a/h \ll |\ln(\bar{\rho}h^2)|^{-1}$, with $\bar{\rho} \sim N/\bar{L}^2$ the mean density, the system is described by a two-dimensional Gross-Pitaevskii density functional with coupling parameter $\sim Na/h$. If $|\ln(\bar{\rho}h^2)|^{-1} \lesssim a/h$ the coupling parameter is $\sim N|\ln(\bar{\rho}h^2)|^{-1}$ and thus independent of a. In both cases Bose-Einstein condensation in the ground state holds, provided the coupling parameter stays bounded.

1 Introduction

In recent experiments dilute Bose gases have been confined in magneto-optical traps in such a way that the particle motion is essentially frozen in one or two directions and the system becomes effectively lower dimensional [8, 9, 33, 20, 23, 10, 31]. This is an intrinsically quantum mechanical phenomenon because it is not necessary to have a trap width or thickness that is the size of an atom, but it suffices that the energy gap for the motion in the strongly confined direction(s) is large compared to the internal energy per particle. The case of highly elongated ("cigar-shaped") traps has received particular attention [21, 4, 2, 3, 7, 26, 15] (see also [15] for further references), because it opens the possibility to realize the one-dimensional Lieb-Liniger model [11], and even the limiting Girardeau-Tonks case [6, 10, 23] that exhibits strong correlations. A detailed, rigorous derivation of the one-dimensional behavior from the many-body Hamiltonian of a three-dimensional gas was given in the paper [15]. This is not a simple problem, one reason being that an approximate factorization of the ground state wave function in the longitudinal and transverse variables is in general not possible (in particular not for hard core potentials), and the proofs in [15] are, in fact, quite long.

In the present paper we carry out a corresponding analysis for thin, disc-shaped traps, i.e., traps with strong confinement in one direction so that a two-dimensional behavior is expected. Experimental realizations of such systems and possible mechanisms for creating them are discussed, e.g., in [8, 9, 1, 38, 31]. On the theoretical side the references [25, 22, 27, 28, 24] contain many valuable insights into their properties. There are several similarities with the emergence of one-dimensional behavior in cigar-shaped traps, but also some notable differences. Like for cigar-shaped traps, there is a basic division of the parameter domain into two regions: one where a limit of a three-dimensional Gross-Pitaevskii (GP) theory applies, and a complementary region described by a "truly" low dimensional theory. In the case discussed in [15] the latter is a density functional theory based on the exact Lieb-Liniger solution for the energy of a strongly interacting (and highly correlated) one-dimensional gas with delta interactions. (Note that in 1D strong interactions means low density.) In the present case, on the other hand, the gas is *weakly* interacting in all parameter regions. In the region not accessible from 3D GP theory the energy formula [32, 19, 14] for a dilute two-dimensional Bose gas with a logarithmic dependence on the density applies. To enter this region extreme dilution is required. The Lieb-Liniger region in the 1D case demands also quite dilute systems, but the requirement is even more stringent in 2D. This will be explained further below.

We recall from [32, 19] that the energy per particle of a dilute, homogeneous, two-dimensional Bose gas with density $\rho_{2\mathrm{D}}$ and scattering length $a_{2\mathrm{D}}$ of the interaction potential is (in units such that $\hbar = 2m = 1$)

$$e_{2\mathrm{D}} \approx 4\pi\rho_{2\mathrm{D}} |\ln(\rho_{2\mathrm{D}} a_{2\mathrm{D}}^2)|^{-1}. \tag{1.1}$$

The corresponding result in three dimensions is

$$e_{3\mathrm{D}} \approx 4\pi\rho_{3\mathrm{D}} a_{3\mathrm{D}}, \tag{1.2}$$

cf. [18]. In the following we shall denote the two-dimensional density, $\rho_{2\mathrm{D}}$, simply by ρ and the three dimensional scattering length, $a_{3\mathrm{D}}$, by a.

Our results establish rigorously, and in the many-body context, a relation between a, the thickness, h, of the trap (assumed to be $\gg a$) and the effective two-dimensional scattering length, $a_{2\mathrm{D}}$. Essentially, as h tends to zero, $a_{2\mathrm{D}} = h\exp(-(\text{const.})h/a)$. (The precise formula is given in Eq. (1.17) below.) If $|\ln \rho h^2| \ll h/a$, then $|\ln(\rho a_{2\mathrm{D}}^2)| \approx h/a$, and the two-dimensional formula (1.1) leads to the same result as the three dimensional formula (1.2), because $\rho_{3\mathrm{D}} \sim \rho/h$. The "true" two dimensional region requires $|\ln \rho h^2| \gtrsim h/a$ and hence the condition $\rho^{-1/2} \gtrsim he^{h/a}$ for the interparticle distance, $\rho^{-1/2}$. This should be compared with the corresponding condition for the 1D Lieb-Liniger region in [15] where the interparticle distance is "only" required to be of the order or larger than h^2/a.

The basic formula $a_{2\mathrm{D}} = h\exp(-(\text{const.})h/a)$ for the scattering length appeared, to the best of our knowledge, first in [25]. It can be motivated by considering a weak, bounded potential, where perturbation theory can be used to compute the energy for a two-body problem that is directly related to the scattering length, cf. Appendix A in [19]. This perturbative calculation is carried out in Section 4 as a step in the proof of a lower bound for the many-body energy; its relation to the formula for $a_{2\mathrm{D}}$ is explained in the Remark after Corollary 4.2. We wish to stress, however, that deriving this formula in the context of two-body scattering is only a step towards the solution of the many-body problem that is the concern of the present paper.

We now define the setting and state the results more precisely. We consider N identical, spinless bosons in a confining, three-dimensional trap potential and with a repulsive, rotationally symmetric pair interaction. We take the direction of strong confinement as the z-direction and

write the points $\mathbf{x} \in \mathbb{R}^3$ as (x, z), $x \in \mathbb{R}^2$, $z \in \mathbb{R}$. The Hamiltonian is

$$H_{N,L,h,a} = \sum_{i=1}^{N} (-\Delta_i + V_{L,h}(\mathbf{x}_i)) + \sum_{1 \le i < j \le N} v_a(|\mathbf{x}_i - \mathbf{x}_j|) \tag{1.3}$$

with

$$V_{L,h}(\mathbf{x}) = V_L(x) + V_h^{\perp}(z) = \frac{1}{L^2} V(L^{-1}x) + \frac{1}{h^2} V^{\perp}(h^{-1}z), \tag{1.4}$$

$$v_a(|\mathbf{x}|) = \frac{1}{a^2} v(a^{-1}|\mathbf{x}|). \tag{1.5}$$

The confining potentials V and $V^{\perp}$ are assumed to be locally bounded and tend to ∞ as $|x|$ and $|z|$ tend to ∞. The interaction potential v is assumed to be nonnegative, of finite range and with scattering length 1; the scaled potential v_a then has scattering length a. We regard v, $V^{\perp}$ and V as fixed and L, h, a as scaling parameters. The Hamiltonian (1.3) acts on symmetric wave functions in $L^2(\mathbb{R}^{3N}, d\mathbf{x}_1 \cdots d\mathbf{x}_N)$. Its ground state energy, $E^{\mathrm{QM}}(N, L, h, a)$, scales with L as

$$E^{\mathrm{QM}}(N, L, h, a) = \frac{1}{L^2} E^{\mathrm{QM}}(N, 1, h/L, a/L). \tag{1.6}$$

Taking $N \to \infty$ but keeping h/L and Na/L fixed leads to a three dimensional Gross-Pitaevskii description of the ground state as proved in [13]. The corresponding energy functional is

$$\mathcal{E}^{\mathrm{GP}}_{3\mathrm{D}}[\phi] = \int_{\mathbb{R}^3} \left\{ |\nabla \phi(\mathbf{x})|^2 + V_{L,h}(\mathbf{x}) |\phi(\mathbf{x})|^2 + 4\pi N a |\phi(\mathbf{x})|^4 \right\} d^3\mathbf{x} \tag{1.7}$$

and the energy per particle is

$$\begin{aligned} E^{\mathrm{GP}}_{3\mathrm{D}}(N, L, h, a)/N &= \inf\{\mathcal{E}^{\mathrm{GP}}_{3\mathrm{D}}[\phi] : \textstyle\int |\phi(\mathbf{x})|^2 d^2\mathbf{x} = 1\} \\ &= (1/L^2) E^{\mathrm{GP}}_{3\mathrm{D}}(1, 1, h/L, Na/L). \end{aligned} \tag{1.8}$$

By Theorem 1.1 in [13], we have, for fixed h/L and Na/L,

$$\lim_{N \to \infty} \frac{E^{\mathrm{QM}}(N, L, h, a)}{E^{\mathrm{GP}}_{3\mathrm{D}}(N, L, h, a)} = 1. \tag{1.9}$$

It is important to note, however, that the estimates in [13] are not uniform in the ratio h/L and the question what happens if $h/L \to 0$ is not addressed in that paper. It will be shown in the next section that a *part* of the parameter range for thin traps can be treated by considering, at fixed Na/h, the $h/L \to 0$ limit of $E^{\mathrm{GP}}_{3\mathrm{D}}(1, 1, h/L, Na/L)$, with the ground state energy for the transverse motion, $\sim 1/h^2$, subtracted. But this limit can evidently never lead to a logarithmic dependence on the density and it does not give the correct limit formula for the energy in the whole parameter range.

To cover all cases we have to consider a two-dimensional Gross-Pitaevskii theory of the type studied in [14], i.e.,

$$\mathcal{E}^{\mathrm{GP}}_{2\mathrm{D}}[\varphi] = \int_{\mathbb{R}^2} \left\{ |\nabla \varphi(x)|^2 + V_L(x) |\varphi(x)|^2 + 4\pi N g |\varphi(x)|^4 \right\} d^2x \tag{1.10}$$

with

$$g = |\ln(\bar{\rho} a^2_{2\mathrm{D}}|^{-1}. \tag{1.11}$$

Here $\bar{\rho}$ is the mean density, defined as in Eq. (1.6) in [14]. An explicit formula that is valid in the case $Ng \gg 1$ can be states as follows. Let

$$\rho_N^{\mathrm{TF}}(x) = \frac{1}{8\pi}\left[\mu_N^{\mathrm{TF}} - V_L(x)\right]_+ \tag{1.12}$$

be the 'Thomas Fermi' density for N particles at coupling constant 1 in the potential V_L, where μ_N^{TF} is chosen so that $\int \rho_N^{\mathrm{TF}} = N$. Then

$$\bar{\rho} = N^{-1}\int_{\mathbb{R}^2} \rho_N^{\mathrm{TF}}(x)^2 dx. \tag{1.13}$$

For simplicity we shall assume that V is homogeneous of some degree $p > 0$, i.e., $V(\lambda x) = \lambda^p V(x)$, and in this case

$$\bar{\rho} \sim N^{p/(p+2)}/L^2 = N/\bar{L}^2 \quad \text{with} \quad \bar{L} = N^{1/(p+2)}L. \tag{1.14}$$

The length $\bar{L}$ measures the effective extension of the gas cloud of the N particles in the two-dimensional trap. A box potential corresponds to $L = \bar{L}$, i.e., $p = \infty$ and hence $\bar{\rho} \sim N/L^2$.

The case $Ng = O(1)$ requires a closer look at the definition of $\bar{\rho}$. First, for any value of Ng we can consider the minimizer $\varphi_{Ng}^{\mathrm{GP}}$ of (1.10) with normalization $\|\varphi_{Ng}^{\mathrm{GP}}\|_2 = 1$. The corresponding mean density is

$$\bar{\rho}_{Ng} = N\int |\varphi_{Ng}^{\mathrm{GP}}|^4. \tag{1.15}$$

A general definition of $\bar{\rho}$ amounts to solving the equation $\bar{\rho} = \bar{\rho}_{Ng}$ with g as in (1.11). As discussed in [14] this gives the same result as (1.13) to leading order in g when $Ng \gg 1$. In the case $|\ln Nh/L| \ll h/a$ (referred to as 'Region I' below) the coupling constant is simply a/h and thus independent of N. Moreover, in a homogeneous potential of degree p the effective length scale $\bar{L}$ is $\sim (Ng+1)^{1/(p+2)}L$ and thus of order L if $Ng = O(1)$.

The energy per particle corresponding to (1.10) is

$$E_{\mathrm{2D}}^{\mathrm{GP}}(N, L, g)/N = \inf\{\mathcal{E}_{\mathrm{2D}}^{\mathrm{GP}}[\varphi] : \int |\varphi(x)|^2 d^2x = 1\} = (1/L^2)E_{\mathrm{2D}}^{\mathrm{GP}}(1, 1, Ng). \tag{1.16}$$

Let s_h be the normalized ground state wave function of the one-particle Hamiltonian $-d^2/dz^2 + V_h^{\perp}(z)$. It can be written as $s_h(z) = h^{-1/2}s(h^{-1}z)$ and the ground state energy as $e_h^{\perp} = h^{-2}e^{\perp}$, where $s(z)$ and $e^{\perp}$ are, respectively, the ground state wave function and ground state energy of $-d^2/dz^2 + V^{\perp}(z)$. We *define* the two dimensional scattering length by the formula

$$a_{\mathrm{2D}} = h\exp\left(-(\textstyle\int s(z)^4 dz)^{-1}h/2a\right). \tag{1.17}$$

Then, using (1.11),

$$g = |-\ln(\bar{\rho}h^2) + (\textstyle\int s(z)^4 dz)^{-1}h/a|^{-1}. \tag{1.18}$$

The justification of the definition (1.17) is Theorem 1.1 below.

Remark. Since a_{2D} appears only under a logarithm, and $a/h \to 0$, one could, at least as far as leading order computations are concerned, equally well define the two dimensional scattering length as $a'_{\mathrm{2D}} = b\exp\left(-(\int s(z)^4 dz)^{-1}h/2a\right)$ with b satisfying $c\,a \le b \le C\,h$ for some constants $c > 0$, $C < \infty$. In fact, if $g' = |\ln(\bar{\rho}(a'_{\mathrm{2D}})^2)|^{-1}$, then

$$\frac{g}{g'} = 1 + \frac{2\ln(b/h)}{|-\ln(\bar{\rho}h^2) + (\text{const.})h/a|} \to 1 \tag{1.19}$$

because $(a/h)\ln(b/h) \to 0$.

We can now state the limit theorem for the ground state energy:

Theorem 1.1 (From 3D to 2D, ground state energy). *Let $N \to \infty$ and at the same time $h/L \to 0$ and $a/h \to 0$ in such a way that $h^2\bar{\rho}g \to 0$ (with g given by Eq. (1.18)). Then*

$$\lim \frac{E^{\mathrm{QM}}(N, L, h, a) - Nh^{-2}e^{\perp}}{E^{\mathrm{GP}}_{\mathrm{2D}}(N, L, g)} = 1. \tag{1.20}$$

Remarks: 1. The condition $h^2\bar{\rho}g \to 0$ means that the ground state energy $h^{-2}e^{\perp}$ associated with the confining potential in the z-direction is much larger than the energy $\bar{\rho}g$. This is the condition of *strong confinement* in the z-direction. In the case that $h/a \gg |\ln(\bar{\rho}h^2)|$ we have $g \sim a/h$ and hence the condition in that region is equivalent to

$$\bar{\rho}ah \ll 1. \tag{1.21}$$

On the other hand, if $h/a \lesssim |\ln(\bar{\rho}h^2)|$ the strong confinement condition is equivalent to $h^2\bar{\rho}|\ln(h^2\bar{\rho})|^{-1} \ll 1$, which means simply that

$$\bar{\rho}h^2 \ll 1\,. \tag{1.22}$$

Both (1.21) and (1.22) clearly imply $\bar{\rho}a_{\mathrm{2D}}^2 \ll 1$, i.e., the gas is dilute in the 2D sense (and also in the 3D sense, $\rho_{\mathrm{3D}}a^3 \ll 1$, because $\rho_{\mathrm{3D}} = \rho/h$). This is different from the situation in cigar-shaped traps considered in [15] where the gas can be either dilute or dense in the 1D sense, depending on the parameters (although it is always dilute in the 3D sense).

2. It is, in fact, not necessary to demand $h/L \to 0$ explicitly in Theorem 1.1. The reason is as follows. In the region where $h/a \lesssim |\ln(\bar{\rho}h^2)|$, the strong confinement condition $\bar{\rho}h^2 \ll 1$ immediately implies $h/L \ll 1$ because $\bar{\rho} \gg 1/L^2$, cf. Eq. (1.14). If $h/a \gg |\ln(\bar{\rho}h^2)|$, then at least $\bar{\rho}ah \ll 1$ holds true. This leaves only the alternatives $h/L \to 0$, or, if h/L stays bounded away from zero, $Na/L \to 0$. But the latter alternative means, by the three dimensional Gross-Pitaevskii limit theorem [13], that the energy converges to the energy of a noninteracting, trapped gas, for which (1.20) obviously holds true.

We shall refer to the parameter region where $h/a \gg |\ln(\bar{\rho}h^2)|$ as **Region I**, and the one where $h/a \lesssim |\ln(\bar{\rho}h^2)|$ as **Region II**. In Region I we can take

$$g = (\textstyle\int s(z)^4 dz)a/h. \tag{1.23}$$

In Region II $g \sim |\ln(\bar{\rho}h^2)|^{-1}$, and in the extreme case that $h/a \ll |\ln(\bar{\rho}h^2)|$,

$$g = |\ln(\bar{\rho}h^2)|^{-1}. \tag{1.24}$$

In particular g is then independent of a (but dependent on $\bar{\rho}$). As remarked earlier, Region II applies only to very dilute gases since it requires interparticle distances $\bar{\rho}^{-1/2} \gtrsim he^{h/a}$.

By Eq. (1.16) the relevant coupling parameter is Ng rather than g itself, and both Region I and Region II can be divided further, according to $Ng \ll 1$, $Ng \sim 1$, or $Ng \gg 1$. The case $Ng \ll 1$ corresponds simply to an ideal gas in the external trap potential. Note that this limit can both be reached from Region I by taking $a/h \to 0$ at fixed $\bar{\rho}h^2$, or from Region II by letting $\bar{\rho}h^2$ tend more rapidly to zero than $e^{-h/a}$. The case $Ng \sim 1$ in Region I corresponds to a GP theory with coupling parameter $\sim Na/h$ as was already explained, in particular after Eq. (1.9). The case $Ng \gg 1$ is the 'Thomas-Fermi' case where the gradient term in the energy functional (1.10) can be ignored. In Region II, the case $Ng \lesssim 1$ requires $\bar{\rho}^{-1/2} \gtrsim he^N$ and is thus mainly of academic interest, while $\bar{\rho}^{-1/2} \ll he^N$ (but still $he^{h/a} \lesssim \bar{\rho}^{-1/2}$) corresponds to the TF case.

The subdivision of the parameter range just described is somewhat different from the situation described in [15]. The reason is the different form of the energy per particle of the low dimensional gas as function of the density.

Theorem 1.1 is a limit theorem for the ground state energy. By standard arguments (variation with respect to the trapping potential) it implies a convergence result for the one particle density in the ground state, $\rho^{\rm QM}_{N,L,h,a}(x)$ (cf. Section 2 in [15] and Theorem 8.2 in [17]):

Define the 2D QM density by integrating over the transverse variable z, i.e.,

$$\hat\rho^{\rm QM}_{N,L,h,a}(x) := \int \rho^{\rm QM}_{N,L,h,a}(x,z)dz \ . \tag{1.25}$$

With $\bar L$ the extension of the system in the 2D trap, cf. (1.14), define the rescaled GP density $\tilde\rho$ by

$$\tilde\rho(x) = \bar L^2 |\varphi^{\rm GP}(\bar L x)|^2 \tag{1.26}$$

where $\varphi^{\rm GP}$ is the minimizer of (1.10) with normalization $\int |\varphi^{\rm GP}(x)|^2 dx = 1$. (Note that $\tilde\rho$ depends on N, L and g.)

Theorem 1.2 (2D limit for the density). *In the same limit as considered in Theorem 1.1,*

$$\lim \left(\frac{\bar L^2}{N} \hat\rho^{\rm QM}_{N,L,h,a}(\bar L x) - \tilde\rho(x) \right) = 0 \tag{1.27}$$

in weak L^1 sense.

In the GP case where the coupling parameter Ng stays bounded a much stronger result can be proved, namely convergence of the 1-particle density matrix and Bose-Einstein condensation (BEC) in the ground state. Recall that the one-body density matrix obtained from the ground state wave function Ψ_0 is

$$\gamma_{\Psi_0}(\mathbf{x},\mathbf{x}') = N \int_{\mathbb{R}^{3(N-1)}} \Psi_0(\mathbf{x},\mathbf{x}_2,\dots,\mathbf{x}_N)\Psi_0(\mathbf{x}',\mathbf{x}_2,\dots,\mathbf{x}_N)^* d\mathbf{x}_2\cdots d\mathbf{x}_N. \tag{1.28}$$

BEC means that in the $N\to\infty$ limit it factorizes as $N\psi(\mathbf{x})\psi(\mathbf{x}')$ for some normalized ψ. This, in fact, is 100% condensation and was proved in [12] for a fixed trap potential in the Gross-Pitaevskii limit, i.e., for both h/L and Na/L fixed as $N\to\infty$. Here we extend this result to the case $h/L\to 0$ with Ng and L fixed. The function ψ is the minimizer of the 2D GP functional (1.10) times the transverse function $s_h(z)$:

Theorem 1.3 (BEC in GP limit). *If $N\to\infty$, $h/L\to 0$ while Ng and L are fixed, then*

$$\frac{h}{N}\gamma_{\Psi_0}(x,hz;x',hz') \to \varphi^{\rm GP}(x)\varphi^{\rm GP}(x')s(z)s(z') \tag{1.29}$$

in trace norm. Here $\varphi^{\rm GP}$ is the normalized minimizer of the GP functional (1.10).

There is a variant of Theorem 1.1 that applies to the thermodynamic limit in the 2D variable x where the density becomes homogeneous in this variable. Let $E_{\rm box}(N,L,a)$ be the ground state energy of the Hamiltonian (1.3) with the potential $V_L(x)$ replaced by a 2D box of side length L and define, for fixed ρ, h and a,

$$e_{\rm 2D}(\rho,h,a) := \lim_{N,L\to\infty, N/L^2=\rho} \frac{E_{\rm box}(N,L,a) - Nh^{-2}e^\perp}{N}. \tag{1.30}$$

This is the energy per particle in the 2D thermodynamic limit (with the confining energy subtracted) and by standard arguments (cf. e.g., [30]) it is independent of the boundary conditions (Dirichlet, Neumann or periodic) imposed in the definition of $E_{\rm box}(N, L, a)$. We then have

Theorem 1.4 (From 3D to 2D, homogeneous case). *If $h^2\rho g \to 0$ and $a/h \to 0$, with g given by Eq.* (1.18)*, then*

$$\lim \frac{e_{\rm 2D}(\rho, h, a)}{4\pi\rho g} = 1. \tag{1.31}$$

The essentials of the dimensional reduction are already contained in the proof of Theorem 1.4 and are more transparent in this case than for inhomogeneous gases in Theorem 1.1. Hence we shall focus on the proof of Theorem 1.4, while the modifications that have to be made for a proof of Theorem 1.1 will be more briefly described with appropriate references to [13, 14, 15, 36], where analogous problems are discussed in some detail.

An abbreviated version of this paper appears as Chapter 9 in the Oberwolfach Seminars volume [17].

2 The 2D limit of 3D GP theory

As in [15], certain aspects of the dimensional reduction of the many-body system can be seen already in the much simpler context of GP theory. We therefore begin by considering the $h/L \to 0$ limit of the 3D GP ground state energy. The result is, apart from the confining energy, the 2D GP energy with coupling constant $g \sim a/h$. This shows in particular that Region II, where $g \sim |\ln(\bar\rho h^2)|^{-1}$, cannot be reached as a limit of 3D GP theory.

Theorem 2.1 (2D limit of 3D GP energy). *Define $g = \left(\int s(z)^4 dz\right) a/h$. If $h/L \to 0$, then*

$$\frac{E^{\rm GP}_{\rm 3D}(N, L, h, a) - Nh^{-2}e^{\perp}}{E^{\rm GP}_{\rm 2D}(N, L, g)} \to 1 \tag{2.1}$$

uniformly in the parameters, as long as $\bar\rho ah \to 0$.

Remark. Since $E^{\rm GP}_{\rm 2D}(1, L, Ng) \sim L^{-2} + \bar\rho a/h$, the condition $\bar\rho ah \to 0$ is equivalent to $h^2 E^{\rm GP}_{\rm 2D}(1, L, Ng) \to 0$, which means simply that the 2D GP energy per particle is much less than the confining energy, $\sim 1/h^2$.

Proof. Because of the scaling relation (1.8) it suffices to consider the case $N = 1$ and $L = 1$.

For an upper bound to the 3D GP ground state energy we make the ansatz

$$\phi(\mathbf{x}) = \varphi^{\rm GP}(x) s_h(z), \tag{2.2}$$

where $\varphi^{\rm GP}$ is the minimizer of the 2D GP functional with coupling constant g. Then

$$\mathcal{E}^{\rm GP}_{\rm 3D}[\phi] = e^{\perp}/h^2 + E^{\rm GP}_{\rm 2D}(1, 1, g) \tag{2.3}$$

and hence

$$E^{\rm GP}_{\rm 3D}(1, 1, h, a) - e^{\perp}/h^2 \leq E^{\rm GP}_{\rm 2D}(1, 1, g). \tag{2.4}$$

For the lower bound we consider the one-particle Hamiltonian (in 3D)

$$H_{h,a} = -\Delta + V_{1,h}(\mathbf{x}) + 8\pi a |\varphi_{\rm GP}(x)|^2 s_h(z)^2. \tag{2.5}$$

Taking the 3D GP minimizer Φ as a test state gives

$$\begin{aligned}\inf\operatorname{spec} H_{h,a} &\leq E^{\mathrm{GP}}_{\mathrm{3D}}(1,1,h,a) - 4\pi a\int_{\mathbb{R}^3}|\Phi(\mathbf{x})|^4 d^3\mathbf{x} \\ &\quad +8\pi a\int |\varphi_{\mathrm{GP}}(x)|^2 s_h(z)^2|\Phi(\mathbf{x})|^2 d^3\mathbf{x} \\ &\leq E^{\mathrm{GP}}_{\mathrm{3D}}(1,1,h,a) + 4\pi a\int_{\mathbb{R}^3}|\varphi^{\mathrm{GP}}(x)|^4 s_h(z)^4 d^3\mathbf{x} \\ &= E^{\mathrm{GP}}_{\mathrm{3D}}(1,1,h,a) + 4\pi g\int_{\mathbb{R}^2}|\varphi^{\mathrm{GP}}(x)|^4 dx. \end{aligned} \tag{2.6}$$

To bound $H_{h,a}$ from below we consider first for fixed $x\in\mathbb{R}^2$ the Hamiltonian (in 1D)

$$H_{h,a,x} = -\partial_z^2 + V_h^\perp(z) + 8\pi a|\varphi^{\mathrm{GP}}(x)|^2 s_h(z)^2. \tag{2.7}$$

We regard $-\partial_z^2 + V_h^\perp(z)$ as its "free" part and $8\pi a|\varphi^{\mathrm{GP}}(x)|^2 s_h(z)^2$ as a perturbation. Since the perturbation is positive all eigenvalues of $H_{h,a,x}$ are at least as large as those of $-\partial_z^2 + V_h(z)$; in particular the first excited eigenvalue is $\sim 1/h^2$. The expectation value in the ground state s_h of the free part is

$$\langle H_{h,a,x}\rangle = e^\perp/h^2 + 8\pi g|\varphi_{\mathrm{GP}}(x)|^2. \tag{2.8}$$

Temple's inequality [37, 29] gives

$$H_{h,a,x} \geq \left(e^\perp/h^2 + 8\pi g|\varphi^{\mathrm{GP}}(x)|^2\right)\left(1 - \frac{\langle (H_{h,a,x} - \langle H_{h,a,x}\rangle)^2\rangle}{\langle H_{h,a,x}\rangle(\tilde{e}^\perp - e^\perp)/h^2}\right), \tag{2.9}$$

where $\tilde{e}^\perp/h^2$ is the lowest eigenvalue above the ground state energy of $-\partial_z^2 + V_h^\perp(z)$. Since

$$H_{h,a,x}s_h = (e^\perp/h^2)s_h + 8\pi a|\varphi^{\mathrm{GP}}(x)|^2 s_h^3 \tag{2.10}$$

we have $(H_{h,a,x} - \langle H_{h,a,x}\rangle)s_h = 8\pi|\varphi_{\mathrm{GP}}(x)|^2(as_h^3 - gs_h)$ and hence, using $g = a\int s_h^4 = (a/h)\int s^4$,

$$\begin{aligned}\langle (H_{h,a,x} - \langle H_{h,a,x}\rangle)^2\rangle &= (8\pi)^2|\varphi_{\mathrm{GP}}(x)|^4\int \left(as_h(z)^3 - gs_h(z)\right)^2 dz \\ &\leq (8\pi)^2\|\varphi^{\mathrm{GP}}\|_\infty^4 (a/h)^2\left[\int s^6 - \left(\int s^4\right)^2\right] \\ &\leq \text{const.}\, E^{\mathrm{GP}}_{\mathrm{2D}}(1,1,g)^2 \end{aligned} \tag{2.11}$$

where we have used Lemma 2.1 in [14] to bound the term $g\|\varphi^{\mathrm{GP}}\|_\infty^2$ by const. $E^{\mathrm{GP}}_{\mathrm{2D}}(1,1,g)$. We thus see from (2.8) and the assumption $h^2 E^{\mathrm{GP}}_{\mathrm{2D}}(1,1,g)\to 0$ that the error term in the Temple inequality (2.9) is $o(1)$.

Now $H_{h,a} = -\Delta_x + V(x) + H_{h,a,x}$, so from (2.9) we conclude that

$$H_{h,a} \geq \left((e^\perp/h^2) - \Delta_x + V(x) + 8\pi g|\varphi^{\mathrm{GP}}(x)|^2\right)(1 - o(1)). \tag{2.12}$$

On the other hand, the lowest energy of $-\Delta_x + V(x) + 8\pi g|\varphi^{\mathrm{GP}}(x)|^2$ is just $E^{\mathrm{GP}}_{\mathrm{2D}}(1,1,g) + 4\pi g\int_{\mathbb{R}^2}|\varphi_{\mathrm{GP}}(x)|^4 dx$. Combining (2.6) and (2.12) we thus get

$$E^{\mathrm{GP}}_{\mathrm{3D}}(1,1,h,a) - e^\perp/h^2 \geq E^{\mathrm{GP}}_{\mathrm{2D}}(1,1,g)(1 - o(1)). \tag{2.13}$$

□

3 Upper bound

3.1 Finite n bounds

We now turn to the many-body problem, i.e., the proof of Theorems 1.1 and 1.4. Like in [15] the key lemmas are energy bounds in boxes with *finite* particle number. The bounds for the total system are obtained by distributing the particles optimally among the boxes. We start with the upper bound for the energy in a single box.

Consider the Hamiltonian

$$H = \sum_{i=1}^{n} \Big(-\Delta_i + V_h^{\perp}(z_i) \Big) + \sum_{1 \le i < j \le n} v_a(|\mathbf{x}_i - \mathbf{x}_j|) \tag{3.1}$$

in a region $\Lambda = \Lambda_2 \times \mathbb{R}$ where Λ_2 denotes a box of side length ℓ in the 2D x variables. For the upper bound on the ground state energy of (3.1) we impose *Dirichlet* boundary conditions on the 2D Laplacian. The goal is to prove, for a given 2D density ρ and parameters a and h, that for a suitable choice of ℓ and corresponding particle number $n = \rho\ell^2$ the energy per particle, with the confining energy $e^{\perp}/h^2$ subtracted, is bounded above by

$$4\pi\rho |\ln(\rho a_{2\mathrm{D}}^2)|^{-1}(1 + o(1)) \tag{3.2}$$

where $a_{2\mathrm{D}}$ is given by Eq. (1.17). Moreover, the Dirichlet localization energy per particle, $\sim 1/\ell^2$, should be small compared to (3.2). The relative error, $o(1)$, in (3.2) tends to zero with the small parameters a/h and $\rho a h$ (Region I), or a/h and ρh^2 (Region II).

The choice of variational functions depends on the parameter regions and we are first concerned with the Region II, i.e., the case $|\ln(\rho h^2)| \gtrsim h/a$.

3.1.1 Upper bound in Region II

Let $f_0(r)$ be the solution of the zero energy scattering equation

$$-\Delta f_0 + \tfrac{1}{2} v_a f_0 = 0, \tag{3.3}$$

normalized so that $f_0(r) = (1 - a/r)$ for $r \ge R_0$ with R_0 the range of v_a. Note that $R_0 = (\text{const.})a$ by the scaling (1.5). The function f_0 satisfies $0 \le f_0(r) \le 1$ and $0 \le f_0'(r) \le \min\{1/r, a/r^2\}$. This is seen by writing $f_0(r) = u(r)/r$ and $f_0'(r) = u'(r)/r - u(r)/r^2$ with $u''(r) = \frac{1}{2} v(r) u(r) \ge 0$. Since $u(r) = r - a$ for $r \ge R_0$ and $u(0) = 0$, convexity implies $u(r) \ge \max\{0, r - a\}$ and $u'(r) \le 1$. Hence $0 \le f_0'(r) \le \min\{1/r, a/r^2\}$ and $0 \le f_0(r) \le \lim_{r\to\infty}(1 - a/r) = 1$.

For $R > R_0$ we define

$$f(r) = f_0(r)/(1 - a/R) \quad \text{for} \quad 0 \le r \le R, \quad \text{and} \quad f(r) = 1 \quad \text{for} \quad r > R. \tag{3.4}$$

We define a two-dimensional potential by

$$W(x) = \frac{2\|s\|_4^4}{h} \int_{\mathbb{R}} \left[f'(|\mathbf{x}|)^2 + \tfrac{1}{2} v_a(|\mathbf{x}|) f(|\mathbf{x}|)^2 \right] dz. \tag{3.5}$$

Clearly, $W(x) \ge 0$, and W is rotationally symmetric with $W(x) = 0$ for $|x| \ge R$. Moreover, by partial integration, using (3.3), it follows that $W \in L^1(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}^2} W(x) dx = \frac{8\pi a \|s\|_4^4}{h} (1 - a/R)^{-1}. \tag{3.6}$$

Define, for $b > R$,

$$\varphi(r) = \begin{cases} \ln(R/a_{2\mathrm{D}})/\ln(b/a_{2\mathrm{D}}) & \text{if} \quad 0 \le r \le R \\ \ln(r/a_{2\mathrm{D}})/\ln(b/a_{2\mathrm{D}}) & \text{if} \quad R \le r \le b \\ 1 & \text{if} \qquad b \le r \end{cases} \tag{3.7}$$

As test function for the three dimensional Hamiltonian (3.1) we shall take

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_n) = F(\mathbf{x}_1, \dots, \mathbf{x}_n) G(\mathbf{x}_1, \dots, \mathbf{x}_n) \tag{3.8}$$

with

$$F(\mathbf{x}_1, \dots, \mathbf{x}_n) = \prod_{i<j} f(|\mathbf{x}_i - \mathbf{x}_j|) \quad \text{and} \quad G(\mathbf{x}_1, \dots, \mathbf{x}_n) = \prod_{i<j} \varphi(|x_i - x_j|) \prod_{k=1}^{n} s_h(z_k). \tag{3.9}$$

The parameters R, b and also ℓ will eventually be chosen so that the errors compared to the expected leading term in the energy are small.

As it stands, the function (3.8) does not satisfy Dirichlet boundary conditions but this can be taken care of by multiplying the function with additional factors at energy cost $\sim 1/\ell^2$ per particle, that will turn out to be small compared to the energy of (3.8).

Since $f(|\mathbf{x}_i - \mathbf{x}_j|)\varphi(|x_i - x_j|) = 1$ for $|x_i - x_j| \ge b$ and s_h is normalized, the norm of Ψ can be estimated as in Eq. (3.9) in [15].

$$\langle \Psi | \Psi \rangle \ge \ell^{2n} \left[1 - \frac{\pi n(n-1)}{2} \frac{b^2}{\ell^2} \right]. \tag{3.10}$$

Next we consider the expectation value of H with the wave function Ψ. By partial integration we have, for every j,

$$\begin{aligned} &\int |\nabla_j (FG)|^2 \\ &= \int G^2 |\nabla_j F|^2 - \int F^2 G \Delta_j G = \int G^2 |\nabla_j F|^2 - \int F^2 G \, \partial_{z_j}^2 G - \int F^2 G (\Delta_j^{\parallel} G) \\ &= \int G^2 |\nabla_j F|^2 - \int F^2 G \, \partial_{z_j}^2 G + \int F^2 |\nabla_j^{\parallel} G|^2 + 2 \int FG (\nabla_j^{\parallel} F) \cdot (\nabla_j^{\parallel} G) \end{aligned} \tag{3.11}$$

where $\Delta_j^{\parallel}$ and $\nabla_j^{\parallel}$ are, respectively, the two dimensional Laplace operator and gradient. The term $-\int F^2 G \, \partial_{z_j}^2 G$ together with $\int V_h^{\perp} F^2 G^2$ gives the confinement energy, $(e^{\perp}/h^2)\|\Psi\|^2$.

Next we consider the first and the third term in (3.11). Since $0 \le f \le 1$, $f' \ge 0$ and s_h is normalized, we have

$$\begin{aligned} \sum_j \int F^2 |\nabla_j^{\parallel} G|^2 + \sum_j \int |\nabla_j F|^2 G^2 \le &\sum_j \int |\nabla_j^{\parallel} \Phi|^2 + 2 \sum_{i<j} f'(|\mathbf{x}_i - \mathbf{x}_j|)^2 G^2 \\ &+ 4 \sum_{k<i<j} \int f'(|\mathbf{x}_k - \mathbf{x}_i|) f'(|\mathbf{x}_k - \mathbf{x}_j|) G^2 \end{aligned} \tag{3.12}$$

where we have denoted $\prod_{i<j} \varphi(|x_i - x_j|)$ by Φ for short. Moreover, since $0 \le \varphi \le 1$,

$$2 \sum_{i<j} f'(|\mathbf{x}_i - \mathbf{x}_j|)^2 G^2 \le 2 \sum_{i<j} f'(|\mathbf{x}_i - \mathbf{x}_j|)^2 s_h(z_i)^2 s_j(z_j)^2. \tag{3.13}$$

By Young's inequality

$$2\int_{\mathbb{R}^2} f'(|\mathbf{x}_i-\mathbf{x}_j|)^2 s_h(z_i)^2 s_j(z_j)^2 dz_i dz_j \leq \frac{2\|s\|_4^4}{h}\int_{\mathbb{R}} f'(|(x_i-x_j,z)|)^2 dz. \tag{3.14}$$

The right side gives rise to the first of the two terms in the formula (3.5) for the two dimensional potential W. The other term is provided by $\int F^2G^2 v_a(\mathbf{x}_i-\mathbf{x}_j)$, using $0\leq f\leq 1$, $0\leq\varphi\leq 1$ and Young's inequality.

Altogether we obtain

$$\langle\Psi|H|\Psi\rangle - (ne^{\perp}/h^2)\langle\Psi|\Psi\rangle \leq \sum_j \int_{\Lambda_2^n} |\nabla_j^{\parallel}\Phi|^2 + \sum_{i<j}\int_{\Lambda_2^n} W(x_i-x_j)\Phi^2 + \mathcal{R}_1+\mathcal{R}_2 \tag{3.15}$$

with

$$\mathcal{R}_1 = 2\int_{\Lambda^n} FG(\nabla_j^{\parallel}F)\cdot(\nabla_j^{\parallel}G) \tag{3.16}$$

and

$$\begin{aligned}\mathcal{R}_2 &= 4\sum_{k<i<j}\int_{\Lambda^n} f'(|\mathbf{x}_k-\mathbf{x}_i|)f'(|\mathbf{x}_k-\mathbf{x}_j|)G^2 \\ &\leq \frac{2}{3}n(n-1)(n-2)\ell^{2(n-3)}\int_{\Lambda^3} f'(|\mathbf{x}_1-\mathbf{x}_2|)f'(|\mathbf{x}_2-\mathbf{x}_3|)s_h(z_1)^2 s_h(z_2)^2 s_h(z_3)^2,\end{aligned} \tag{3.17}$$

where $0\leq\varphi\leq 1$ has been used for the last inequality.

The error term $\mathcal{R}_1$ is easily dealt with: It is zero because $\varphi(r)$ is constant for $r\leq R$ and $f(r)$ is constant for $r\geq R$.

The other term, $\mathcal{R}_2$, is estimated as follows. Since $f'(r)=0$ for $r\geq R$ we can use the Cauchy Schwarz inequality for the integration over $\mathbf{x}_1$ at fixed $\mathbf{x}_2$ to obtain

$$\begin{aligned}&\int f'(|\mathbf{x}_1-\mathbf{x}_2|)s_h(z_1)^2 d\mathbf{x}_1 \\ &\leq \left(\int f'(|\mathbf{x}_1-\mathbf{x}_2|)^2 d\mathbf{x}_1\right)^{1/2}\left(\int_{|\mathbf{x}_1-\mathbf{x}_2|\leq R} s_h(z_1)^4 d\mathbf{x}_1\right)^{1/2} \\ &\leq (4\pi\|s\|_\infty a' R^3/3h^2)^{1/2}\end{aligned} \tag{3.18}$$

with $a'=a(1-a/R)^{-1}$. The same estimate for the integration over $\mathbf{x}_3$ and a subsequent integration over $\mathbf{x}_2$ gives

$$\mathcal{R}_2 \leq (\text{const.})\ell^{2n}n^3\frac{a'R^3}{\ell^4h^2}. \tag{3.19}$$

We need $\mathcal{R}_2/\langle\Psi|\Psi\rangle$ to be small compared to the leading term in the energy, $\sim n^2\ell^{-2}|\ln(\rho h^2)|^{-1}$ with $\rho=n/\ell^2$. (Recall that we are in Region II where $|\ln(\rho h^2)|\gtrsim h/a$.) Moreover, the leading term should be large compared to the Dirichlet localization energy, which is $\sim n/\ell^2$. We are thus lead to the conditions (the first comes from (3.10)):

$$\frac{n^2b^2}{\ell^2}\ll 1,\quad \frac{na'R^3|\ln(\rho h^2)|}{\ell^2h^2}\ll 1,\quad \frac{n}{|\ln(\rho h^2)|}\gg 1, \tag{3.20}$$

which can also be written

$$\rho^2\ell^2b^2\ll 1,\quad \frac{\rho a'R^3|\ln(\rho h^2)|}{h^2}\ll 1,\quad \frac{|\ln(\rho h^2)|}{\rho\ell^2}\ll 1. \tag{3.21}$$

These conditions are fulfilled if we choose

$$R = h, \quad b = \rho^{-1/2} |\ln(\rho h^2)|^{-\alpha} \tag{3.22}$$

with $\alpha > 1/2$ and

$$\rho^{-1/2} |\ln(\rho h^2)|^{1/2} \ll \ell \ll \rho^{-1/2} |\ln(\rho h^2)|^{\alpha}. \tag{3.23}$$

Note also that $n = \rho \ell^2 \gg 1$.

It remains to compare

$$\langle \Psi | \Psi \rangle^{-1} \left(\sum_j \int_{\mathbb{R}^{2n}} |\nabla_j^{\parallel} \Phi|^2 + \sum_{i<j} \int_{\mathbb{R}^{2n}} W(x_i - x_j) \Phi^2 \right) \tag{3.24}$$

with the expected leading term of the energy, i.e., $4\pi (n^2/\ell^2) |\ln(n a_{2\mathrm{D}}^2/\ell^2)|^{-1}$.

We consider first the simplest case, i.e., $n = 2$. We have

$$\int_{\mathbb{R}^2} |\nabla^{\parallel} \varphi|^2 = (\ln(b/a_{2\mathrm{D}}))^{-2} 2\pi \int_R^b \frac{dr}{r} = (\ln(b/a_{2\mathrm{D}}))^{-2} 2\pi \ln(b/R), \tag{3.25}$$

$$\tfrac{1}{2} \int_{\mathbb{R}^2} W \varphi^2 = \frac{4\pi a \|s\|_4^4}{h} \left(\frac{\ln(R/a_{2\mathrm{D}})}{\ln(b/a_{2\mathrm{D}})} \right)^2. \tag{3.26}$$

Inserting the formula (1.17) for $a_{2\mathrm{D}}$ and using $R = h$, $b = \rho^{-1/2} |\ln(\rho h^2)|^{-\alpha}$ and $a' = a(1 + o(1))$ we have

$$\int_{\mathbb{R}^2} (|\nabla^{\parallel} \varphi|^2 + \tfrac{1}{2} W \varphi^2) = 2\pi (\ln(b/a_{2\mathrm{D}}))^{-2} \left[\ln(b/h) + (h/2a' \|s\|_4^4) \right] = 4\pi |\ln(\rho a_{2\mathrm{D}}^2)|^{-1} (1 + o(1)). \tag{3.27}$$

For $n > 2$ we can use the symmetry of Φ to write, using (3.27) as well as $0 \leq \varphi(r) \leq 1$,

$$\begin{aligned} &\sum_j \int_{\Lambda_2^n} |\nabla_j^{\parallel} \Phi|^2 + \sum_{i<j} \int_{\Lambda_2^n} W(x_i - x_j) \Phi^2 \\ &= n \left(\int_{\Lambda_2^n} |\nabla_1^{\parallel} \Phi|^2 + \tfrac{1}{2} \sum_{i=2}^n \int_{\Lambda_2^n} W(x_i - x_1) \Phi^2 \right) \\ &\leq 4\pi n^2 \ell^{2(n-1)} |\ln(n a_{2\mathrm{D}}^2/\ell^2)|^{-1} (1 + o(1)) + \mathcal{R}_3 \end{aligned} \tag{3.28}$$

with

$$\mathcal{R}_3 = n^3 \ell^{2(n-3)} \int_{\Lambda_2^3} \varphi'(|x_2 - x_1|) \varphi'(|x_3 - x_1|). \tag{3.29}$$

We estimate $\mathcal{R}_3$ in the same way as (3.18), obtaining

$$\mathcal{R}_3 \leq (\text{const.}) \ell^{2(n-2)} n^3 b^2 (\ln(b/a_{2\mathrm{D}}))^{-2} 2\pi \ln(b/R). \tag{3.30}$$

The condition that $\mathcal{R}_3$ has to be much smaller than the leading term, given by $4\pi n^2 \ell^{2(n-1)} |\ln(n a_{2\mathrm{D}}^2/\ell^2)|^{-1}$, is equivalent to

$$\frac{n b^2}{\ell^2} \ln(b/R) \ll 1. \tag{3.31}$$

With the choice (3.22) this holds if $\alpha > 1/2$.

3.1.2 Upper bound in Region I

In Region I the ansatz (3.8) can still be used, but this time we take $b = R$, i.e., $\varphi \equiv 1$. In this region $(a/h)|\ln(\rho h^2)| = o(1)$ and the leading term in the energy is $\sim n^2\ell^{-2}a/h$. Conditions (3.20) are now replaced by

$$\frac{n^2R^2}{\ell^2} \ll 1, \quad \frac{nR^3}{\ell^2 h} \ll 1, \quad \frac{na}{h} \gg 1 \tag{3.32}$$

where have here used that $a' = a(1 + o(1))$, provided $R \gg a$. Note that the last condition in (3.32) means in particular that $n \gg 1$. Putting again $\rho = n/\ell^2$, (3.32) can be written as

$$\rho^2\ell^2R^2 \ll 1, \quad \frac{\rho R^3}{h} \ll 1, \quad \frac{h}{\rho\ell^2 a} \ll 1. \tag{3.33}$$

By assumption, $a/h \ll 1$, but also $\rho ah \ll 1$ by the condition of strong confinement, c.f. (1.21). We take

$$R = a(\rho ah)^{-\beta} \tag{3.34}$$

with $0 < \beta$, so $R \gg a$. Further restrictions come from the conditions (3.33): The first and the last of these conditions imply together that

$$\frac{h}{a} \ll \rho\ell^2 \ll \frac{1}{\rho R^2} \tag{3.35}$$

which can be fulfilled if

$$\rho ah \ll (\rho ah)^{2\beta}. \tag{3.36}$$

i.e., if $\beta < \frac{1}{2}$. Note that this implies in particular $R \ll \rho^{-1/2}$. We can then take

$$\ell = \rho^{-1/2}(h/a)^{1/2}(\rho ah)^{-\gamma} \tag{3.37}$$

with

$$0 < \gamma < \frac{1 - 2\beta}{2}. \tag{3.38}$$

The second of the conditions (3.33) requires that

$$\frac{\rho R^3}{h} = (\rho ah)(a/h)^2(\rho ah)^{-3\beta} \ll 1, \tag{3.39}$$

which holds in any case if $\beta \leq 1/3$. A possible choice satisfying all conditions is

$$\beta = \frac{1}{3}, \quad \gamma = \frac{1}{12}. \tag{3.40}$$

The error terms (3.33) are then bounded by $(\rho ah)^{1/6}$ (first and third term) and $(a/h)^2$ (second term).

Finally, with $\Phi \equiv 1$, Eqs. (3.15), (3.10) and (3.6) give

$$\langle\Psi|\Psi\rangle^{-1}\langle\Psi|H|\Psi\rangle - (ne^{\perp}/h^2) \leq \frac{4\pi n^2}{\ell^2}\frac{a\|s\|_4^4}{h}(1 + o(1)). \tag{3.41}$$

This completes the proof of the upper bound in boxes with finite n.

3.2 Global bound, uniform case

The upper bound for the energy per particle in the 2D thermodynamic limit needed for Theorem 1.4 is now obtained by dividing $\mathbb{R}^2$ into Dirichlet boxes with side length ℓ satisfying (3.23) in Region II, or (3.35) in Region I, and distributing the particles evenly among the boxes. In other words, the trial wave function in a large box of side length L is

$$\Psi = \sum_\alpha \Psi_\alpha \tag{3.42}$$

where α labels the boxes of side length ℓ contained in the large box, and Ψ_α is the Dirichlet ground state wave function for $n = \rho\ell^2$ particles in the box α, with $\rho = N/L^2$. The choice of ℓ guarantees in particular that the error associated with the Dirichlet localization is negligible. In order to avoid contributions from the interaction between particles in different boxes the boxes should be separated by the range, R_0 of the interaction potential and in the "corridors" between the boxes the wave function is put equal to zero. The number of particles in each box is then not exactly $\rho\ell^2$, but rather the smallest integer larger than $\rho\ell^2(\ell/(\ell+R_0))^2$. In order that the errors are negligible one needs $R_0/\ell = o(1)$, as well as $\rho\ell^2 \gg 1$, and both are garanteed by the choice (3.37) of ℓ.

3.3 Global bound in a trap

In Theorem 1.1 the system is inhomogeneous in 2D because of the trapping potential $V_L(x)$, and the distribution of the particles among the boxes has to be adjusted to the density given by the GP minimizer $\varphi^{\mathrm{GP}}(x)$. As long as ℓ, given by (3.37) or (3.23), is small compared to L this can be done in an analogous way as in [15], Section 4.2. Since, however, the condition $\ell \ll L$ requires $Na/h \gg (\bar\rho a h)^{-1/12}$ in Region I, and $N/|\ln(\bar\rho h^2)| \gg |\ln(\bar\rho h^2)|^\varepsilon$ in Region II, this will not work in the whole parameter range.

A better and only slightly more complicated choice that works in all cases is to replace the Jastrow-type function $\prod_{i<j} f(|\mathbf{x}_i - \mathbf{x}_j|)$ by a Dyson wave function of the type considered in [5] and [13], i.e.,

$$F(\mathbf{x}_1, \dots, \mathbf{x}_N) = \prod_{i=1}^N f(t_i(\mathbf{x}_1, \dots, \mathbf{x}_i)) \tag{3.43}$$

where

$$t_i(\mathbf{x}_1, \dots, \mathbf{x}_i) = \min_{1\le j\le i-1} |\mathbf{x}_i - \mathbf{x}_j| \tag{3.44}$$

is the distance between $\mathbf{x}_i$ and its nearest neighbor among the points $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}$. To take care of the inhomogeneity in the 2D variables a factor $\prod_i \varphi^{\mathrm{GP}}(x_i)$ is included, where φ^{GP} the normalized minimizer of the 2D GP functional with coupling constant Ng. Moreover, the local behavior in the 2D variables is modelled by a Dyson wave function

$$\Phi(x_1, \dots, x_N) = \varphi(t_i'(x_1, \dots, x_N)) \tag{3.45}$$

with

$$t_i'(x_1, \dots, x_i) = \min_{1\le j\le i-1} |x_i - x_j|. \tag{3.46}$$

The global trial wave function is thus of the form

$$\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) = F(\mathbf{x}_1, \dots, \mathbf{x}_N) G(\mathbf{x}_1, \dots, \mathbf{x}_N) \tag{3.47}$$

with F given by (3.43), and

$$G(\mathbf{x}_1,\dots,\mathbf{x}_N)=\Phi(x_1,\dots,x_N)\prod_i \varphi^{\mathrm{GP}}(x_i)s_h(z_i). \tag{3.48}$$

The advantage of the Dyson wave functions over the Jastrow-type functions is that when computing the expectation values $\langle\Psi,H\Psi\rangle/\langle\Psi,\Psi\rangle$ cancellations between numerator and denominator take place that effectively improve the estimate (3.10) of the norm of Ψ.

3.3.1 Region I

In Region I, where one can take $\varphi\equiv 1$, the computation of the expectation value of the Hamiltonian is the same as in [13]. In fact, there is even no need to do the computations explicitly because the required bound can be obtained by combining Theorem 2.1 with Theorem III.1 in [13]. One checks that the parameter b in Eq. (3.42) in [13] is proportional to $\bar\rho^{-1/3}h^{1/3}$ and hence the error terms in the upper bound of E^{QM} in terms of the 3D GP energy are of the order $a/b=(a/h)^{2/3}(\bar\rho ah)^{1/3}$. On the other hand, by Theorem 2.1 the 3D GP energy minus confining energy approaches, as $h/L\to 0$, the 2D GP energy with coupling constant $(\int s^4)a/h$ and the approximation is uniform in the parameters as long as $(\bar\rho ah)\to 0$. Altogether we have the bound

$$E^{\mathrm{QM}}(N,L,h,a)-Nh^{-2}e^{\perp}\le E^{\mathrm{GP}}_{\mathrm{2D}}(N,L,(\textstyle\int s^4)a/h)(1+o(1)) \tag{3.49}$$

where the error term o(1) tend to zero if $h/L\to 0$, $a/h\to 0$ and $\bar\rho ah\to 0$.

3.3.2 Region II

In Region II one uses the ansatz (3.47)–(3.48) with f as in (3.4) with $R=h$ and φ as in (3.7) with $b=\bar\rho^{-1/2}|\ln(\bar\rho h^2)|^{\alpha}$. Since the computations are analogous to those in [13] and in Eqs. (3.11)-(3.29) above we shall not write all details explicitly. Proceeding as in Section 3.1 in [13], using (3.11), we obtain for the expectation value of the Hamiltonian (1.3)

$$\begin{aligned}\frac{\langle\Psi,H^{(N)}\Psi\rangle}{\langle\Psi|\Psi\rangle}\le{}& 2\sum_{i=1}^N\frac{\int|\Psi|^2F_i^{-2}f'(t_i)^2}{\int|\Psi|^2}+\sum_{j<i}\frac{\int|\Psi|^2v(|x_i-x_j|)}{\int|\Psi|^2}\\ &+2\sum_{k\le i<j}\frac{\int|\Psi|^2|\varepsilon_{ik}\varepsilon_{jk}|F_i^{-1}F_j^{-1}f'(t_i)f'(t_j)}{\int|\Psi|^2}\\ &+\sum_{i=1}^N\frac{\int F^2G(-\nabla_i^2G)+|\Psi|^2V_{L,h}(\mathbf{x}_i)}{\int|\Psi|^2}.\end{aligned} \tag{3.50}$$

where

$$\varepsilon_{ik}=\begin{cases}1 & \text{for } i=k\\ -1 & \text{for } t_i=|x_i-x_k|\\ 0 & \text{otherwise}\end{cases}. \tag{3.51}$$

The next step is to decouple a given pair of variables, $\mathbf{x}_i$, $\mathbf{x}_j$, from the other variables in order to achieve cancellations between numerator and denominator in the terms on the right side of (3.50). As in [13] one defines for $i<j<p$

$$F_{p,i}=f(\min_{k<p,k\neq i}|\mathbf{x}_p-\mathbf{x}_k|),\qquad F_{p,ij}=f(\min_{k<p,k\notin\{i,j\}}|\mathbf{x}_p-\mathbf{x}_k|). \tag{3.52}$$

Note that $F_{p,i}$ is independent of i and $F_{p,ij}$ is independent of i and j. As in Eqs. (3.13)–(3.15) in [13] one has

$$F_{j+1}^2 \dots F_{i-1}^2 F_{i+1}^2 \dots F_N^2 \leq F_{j+1,j}^2 \dots F_{i-1,j}^2 F_{i+1,ij}^2 \dots F_{N,ij}^2 \tag{3.53}$$

and

$$\begin{aligned} F_j^2 \dots F_N^2 &\geq F_{j+1,j}^2 \dots F_{i-1,j}^2 F_{i+1,ij}^2 \dots F_{N,ij}^2 \\ &\times \left(1 - \sum_{k=1,\, k\neq i,j}^{N} (1 - f(|\mathbf{x}_j - \mathbf{x}_k|)^2)\right) \left(1 - \sum_{k=1,\, k\neq i}^{N} (1 - f(|\mathbf{x}_i - \mathbf{x}_k|)^2)\right). \end{aligned} \tag{3.54}$$

In an analogous way one defines $\Phi_{p,i}$ and $\Phi_{p,ij}$ and obtains the corresponding equations (3.53) and (3.54) with f replaced by φ and $\mathbf{x}$ by the 2D variable x.

Consider now the first two terms in (3.50). As in [13] one uses the estimates

$$f'(t_i)^2 \leq \sum_{j=1}^{i-1} f'(|\mathbf{x}_i - \mathbf{x}_j|)^2, \quad \text{and} \quad F_i \leq f(|\mathbf{x}_i - \mathbf{x}_j|). \tag{3.55}$$

For fixed i, j one can use (3.53) and (3.54) (and their analogues for Φ) to separate the contribution from the variables $\mathbf{x}_i$ and $\mathbf{x}_j$ from the rest of the integrand. The part that depends only on the other variables cancels between numerator and denominator, while the ij contribution in the numerator is

$$\begin{aligned} \int \big(2f'(|\mathbf{x}_i - \mathbf{x}_j|)^2 + v(|\mathbf{x}_i - \mathbf{x}_j|) f(|\mathbf{x}_i - \mathbf{x}_j|)^2\big) \\ \times s_h(z_i)^2 s_h(z_j)^2 \varphi(|(x_i - x_j|)^2 \varphi^{\mathrm{GP}}(x_i)^2 \varphi^{\mathrm{GP}}(x_j)^2 d\mathbf{x}_i d\mathbf{x}_j. \end{aligned} \tag{3.56}$$

Integration over z_i and z_j, using Young's inequality, bounds (3.56) by

$$\int W(x_i - x_j) \varphi(|(x_i - x_j|) \varphi^{\mathrm{GP}}(x_i)^2 \varphi^{\mathrm{GP}}(x_j)^2. \tag{3.57}$$

where W is defined in (3.5). This, in turn, can as in Eqs. (3.18)–(3.20) in [13] be bounded by

$$2 \int \varphi^{\mathrm{GP}}(x)^4 dx J \tag{3.58}$$

with

$$J = \tfrac{1}{2} \int W(x) \varphi(|x|)^2 dx \tag{3.59}$$

given by (3.26) The matching kinetic term, $\int |\nabla^{\|} \varphi|^2$, given by (3.25) ist derived form the last term in (3.50) in the same way. These two terms together give, up to small errors, the correct coupling constant, i.e., the factor before $\int \varphi^{\mathrm{GP}}(x)^4 dx$ in the 2D GP functional. The other parts of the 2D GP functional as well as the confining energy $Nh^{-2}e^{\perp}$ follow from the last term in (3.50).

It remains to look at the error terms. These come on the one hand from the ij contribution to the denominator, i.e., the last two factors in (3.54) and the corresponding factors with φ instead of f. On the other hand they come from the third term in (3.50) and corresponding

terms with φ instead of f. The first mentioned errors have (cf. Eq. (3.21) in [15] and Eq. (3.1) in [14]) the form

$$N\|\varphi^{\mathrm{GP}}\|_\infty^2\|s_h\|_\infty^2(4\pi R^3/3), \quad \text{and} \quad N\|\varphi^{\mathrm{GP}}\|_\infty^2\pi b^2. \tag{3.60}$$

Since $\|\varphi^{\mathrm{GP}}\|_\infty^2 \sim \bar{L}^{-2}$ and $\|s_h\|_\infty^2 \sim h^{-1}$, while $R = h$ and $b = \bar{\rho}^{-1/2}|\ln(\bar{\rho}h^2)|^{-\alpha}$, these terms are of the order $\bar{\rho}h^2$ and $|\ln(\bar{\rho}h^2)|^{-2\alpha}$ respectively. For the other errors one again exploits the cancellations between numerator and denominator and obtains in the same way as in [15] and [14] (cf. Eqs. (3.26) and (3.36) in [13], and (3.4)–(3.5) in [14]) terms of the order $(\bar{\rho}ah)^2$, N^{-1}, and $|\ln(\bar{\rho}h^2)|^{-1}$. This can be compared with the estimate for $\mathcal{R}_2$ in (3.17) where cancellations could not be used. All the mentioned error terms are small in Region II. In combination with the bound (3.49) for Region I we can thus state the upper bound in both regions as

$$E^{\mathrm{QM}}(N, L, h, a) - Nh^{-2}e^{\perp} \leq E_{\mathrm{2D}}^{\mathrm{GP}}(N, L, g)(1 + o(1)). \tag{3.61}$$

4 Scattering length

As a preparation for the lower bound we consider in this section the perturbative calculation of the 2D scattering length of an integrable potential.

Consider a 2D, rotationally symmetric potential $W \geq 0$ of finite range R_0. As discussed in Appendix A in [19] the scattering length is determined by minimizing, for $R \geq R_0$, the functional

$$\mathcal{E}_R[\psi] = \int_{|x|\leq R} \left\{|\nabla\psi|^2 + \tfrac{1}{2}W|\psi|^2\right\} \tag{4.1}$$

with boundary condition $\psi = 1$ for $|x| = R$. The Euler equation (zero energy scattering equation) is

$$-\Delta\psi + \tfrac{1}{2}W\psi = 0 \tag{4.2}$$

and for $r = |x| \geq R_0$ the minimizer, ψ_0, is

$$\psi_0(r) = \ln(r/a_{\mathrm{scatt}})/\ln(R/a_{\mathrm{scatt}}) \tag{4.3}$$

with a constant a_{scatt}. This is, by definition, the 2D scattering length for the potential W. An equivalent definition follows by computing the energy,

$$E_R = \mathcal{E}_R[\psi_0] = 2\pi/\ln(R/a_{\mathrm{scatt}}) \tag{4.4}$$

which means that

$$a_{\mathrm{scatt}} = R\exp(-2\pi/E_R). \tag{4.5}$$

Lemma 4.1 (Scattering length for soft potentials). *Assume $W(x) = \lambda w(x)$ with $\lambda \geq 0$, $w \geq 0$, and $w \in L^1(\mathbb{R}^2)$, with $\int w(x)dx = 1$. Then, for $R \geq R_0$,*

$$a_{\mathrm{scatt}} = R\exp\left(-\frac{4\pi + \eta(\lambda, R)}{\lambda}\right) \tag{4.6}$$

with $\eta(\lambda, R) \to 0$ for $\lambda \to 0$.

Proof. The statement is, by (4.5), equivalent to

$$E_R = \tfrac{1}{2}\lambda(1 + o(1)) \tag{4.7}$$

where the error term may depend on R. The upper bound is clear by the variational principle, taking $\psi = 1$ as a test function. For the lower bound note first that $\psi_0 \leq 1$. This follows from the variational principle: Since $W \geq 0$ the function $\tilde{\psi}_0(x) = \min\{1, \psi_0\}$ satisfies $\mathcal{E}_R[\tilde{\psi}_0] \leq \mathcal{E}_R[\psi_0]$. Hence the function $\varphi_0 = 1 - \psi_0$ is nonnegative. It satisfies

$$-\Delta\varphi_0 + \tfrac{1}{2}W\varphi_0 = \tfrac{1}{2}W \tag{4.8}$$

and the Dirichlet boundary condition $\varphi_0 = 0$ for $|x| = R$.

Integration of (4.2), using that $\psi_0(r) = 1$ for $r = R$, gives

$$E_R = \tfrac{1}{2}\int W\psi_0 = \tfrac{1}{2}\int W(1 - \varphi_0). \tag{4.9}$$

Since $\varphi_0 \geq 0$ we thus need to show that $\|\varphi_0\|_\infty = o(1)$.

By (4.8), and since φ_0 and W are both nonnegative, we have $-\Delta\varphi_0 \leq \frac{1}{2}W$ and hence

$$\varphi_0(x) \leq \int K_0(x, x')W(x')dx' \tag{4.10}$$

where $K_0(x, x')$ is the (nonnegative) integral kernel of $(-\Delta)^{-1}$ with Dirichlet boundary conditions at $|x| = R$. The kernel $K_0(x, x')$ is integrable (the singularity is $\sim \ln|x - x'|$) and hence, if W is bounded, we have $\|\varphi_0\|_\infty \leq (\text{const.})\lambda\|w\|_\infty = O(\lambda)$.

If w is not bounded we can, for every $\varepsilon > 0$, find a bounded $w^\varepsilon \leq w$ with $\int(w - w^\varepsilon) \leq \varepsilon$. Define $C_\varepsilon = \|w^\varepsilon\|_\infty$. Without restriction we can assume that C_ε is a monotonously decreasing function of ε and continuous. The function $g(\varepsilon) = \varepsilon/C_\varepsilon$ is then monotonously increasing in ε (and hence decreasing if ε decreases), continuous and $g(\varepsilon) \to 0$ if $\varepsilon \to 0$. For every (sufficiently small) λ there is an $\varepsilon(\lambda) = o(1)$ such that $g(\varepsilon(\lambda)) = \lambda$. Then

$$\|\varphi_0\|_\infty \leq (\text{const.})(\varepsilon(\lambda) + \lambda C_{\varepsilon(\lambda)}) = (\text{const.})\varepsilon(\lambda) = o(1). \tag{4.11}$$

□

Corollary 4.2 (Scattering length for scaled, soft potentials). *Assume* $W_{R,\lambda}(x) = \lambda R^{-2}w_1(x/R)$ *with* $w_1 \geq 0$ *fixed and* $\int w_1 = 1$. *Then the scattering length of* $W_{R,\lambda}$ *is*

$$a_{\text{scatt}} = R\exp\left(-\frac{4\pi + \eta(\lambda)}{\lambda}\right) \tag{4.12}$$

with $\eta(\lambda) \to 0$ *for* $\lambda \to 0$, *independent of* R.

Proof. This follows from Lemma 4.1 noting that, by scaling, the scattering length of $W_{R,\lambda}$ is R times the scattering length of λw_1. The latter is independent of R. □

Remark. If W is obtained by averaging a 3D integrable potential v over an interval of length h in the z variable, the formula (4.12), together with Eq. (A.8) in [19], motivates the exponential dependence of the effective 2D scattering length (1.17) of v on h/a: The integral $\lambda = \int W(x)dx$ is $h^{-1}\int v(\mathbf{x})d^3\mathbf{x}$, which for weak potentials is $h^{-1}8\pi a$ to lowest order, by Eq. (A.8) in [19]. Inserting this into (4.12) gives (1.17) (apart from the dependence on the shape function s). This heuristics is, of course, only valid for soft potentials v. An essential step in the lower bound in the next section is the replacement of v by a soft potential to which this reasoning can indeed be applied.

5 Lower bound

5.1 Finite boxes

Like for the upper bound we consider first the homogeneous case and finite boxes, this time with Neumann boundary conditions. The optimal distribution of particles among the boxes is determined by using subadditivity and convexity arguments as in [18] and [15].

In the treatment of the lower bound there is a natural division line between the case where the mean particle distance $\rho^{-1/2}$ is comparable to or larger than h and the case where it is much smaller than h. The first case includes Region II and a part (but not all) of Region I. When $\rho^{-1/2}$ is much smaller than h the boxes have finite extension in the z direction as well. The method is then a fairly simple modification of the 3D estimates in [13] (see also Section 4.4 in [15]) and will not be discussed further here.

The derivation of a lower bound for the case that $\rho h^2 \leq C < \infty$ proceeds by the following steps:

- Use Dyson's Lemma [5, 18] to replace v_a by an integrable 3D potential U, retaining part of the kinetic energy.
- Average the potential U at fixed $x \in \mathbb{R}^2$ over the z-variable to obtain a 2D potential W. Estimate the error in this averaging procedure by using Temple's inequality [37, 29] at each fixed x.
- The result is a 2D many body problem with an integrable interaction potential W which, by Corollary 4.2, has the right 2D scattering length to lowest order in a/h, but reduced kinetic energy inside the range of the potential. This problem is treated in the same way as in [19, 14], introducing a 2D Dyson potential and using perturbation theory, again estimating the errors by Temple's inequality.
- Choose the parameters (size ℓ of box, fraction ε of the kinetic energy, range R of potential U, as well as the corresponding parameters for the 2D Dyson potential) optimally to minimize the errors.

The first two steps are analogous to the corresponding steps in the proof of the lower bound in Theorem 3.1 in [15], cf. Eqs. (3.30)–(3.36) in [15]. It is, however, convenient to define the Dyson potential U in a slightly different manner than in [18]. Namely, for $R \geq 2R_0$, with R_0 the range of v_a, we define

$$U_R(r) = \begin{cases} \frac{24}{7} R^{-3} & \text{for } \frac{1}{2}R < r < R \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The reason is that this potential has a simple scaling with R which is convenient when applying Corollary 4.2. Proceeding as in Eqs. (3.30)–(3.38) in [15] we write a general wave function as

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \prod_{k=1}^{n} s_h(z_k) \ , \tag{5.2}$$

and define $F(x_1, \ldots, x_n) \geq 0$ by

$$|F(x_1, \ldots, x_n)|^2 = \int |f(\mathbf{x}_1, \ldots, \mathbf{x}_n)|^2 \prod_{k=1}^{n} s_h(z_k)^2 dz_k \ . \tag{5.3}$$

Note that F is normalized if Ψ is normalized. The analogue of Eq. (3.38) in [15] is

$$\begin{aligned}&\langle\Psi|H|\Psi\rangle - \frac{ne^\perp}{h^2} \geq \\ &\sum_{i=1}^n \int \left[\varepsilon|\nabla_i^\parallel F|^2 + (1-\varepsilon)|\nabla_i^\parallel F|^2 \chi_{\min_k |x_i - x_k| \geq R}(x_i)\right] \prod_{k=1}^n dx_k \\ &+ \sum_{i=1}^n \int \left[\varepsilon|\partial_i f|^2 + a' U_R(|\mathbf{x}_i - \mathbf{x}_{k(i)}|)\chi_{\mathcal{B}_\delta}(z_{k(i)}/h)|f|^2\right] \prod_{k=1}^n s_h(z_k)^2 d\mathbf{x}_k,\end{aligned} \tag{5.4}$$

where $\nabla_i^\parallel$ denotes the gradient with respect to x_i and $\partial_j = d/dz_j$. Moreover, $\chi_{\mathcal{B}_\delta}$ is the characteristic function of the subset $\mathcal{B}_\delta \subset \mathbb{R}$ where $s(z)^2 \geq \delta$ for $\delta > 0$,

$$a' = a(1-\varepsilon)(1 - 2R\|\partial s^2\|_\infty/(h\delta)), \tag{5.5}$$

and $k(i)$ denotes the index of the nearest neighbor to $\mathbf{x}_i$. When deriving (5.4) the Cauchy Schwarz inequality has been used to bound the longitudinal kinetic energy of f in terms of that of F, i.e.,

$$\sum_{i=1}^n \int |\nabla_i^\parallel f|^2 \prod_{k=1}^n s_h(z_k)^2 d\mathbf{x}_k \geq \sum_{i=1}^n \int |\nabla_i^\parallel F|^2 \prod_{k=1}^n dx_k. \tag{5.6}$$

We now consider, for fixed $x_1, \ldots, x_n$, the term

$$\sum_{i=1}^n \int \left[\varepsilon|\partial_i f|^2 + a' U_R(|\mathbf{x}_i - \mathbf{x}_{k(i)}|)\chi_{\mathcal{B}_\delta}(z_{k(i)}/h)|f|^2\right] \prod_{k=1}^n s_h(z_k)^2 dz_k \ . \tag{5.7}$$

This can be estimated from below by the expectation value of U_R over z_i at fixed x_i, using Temple's inequality to estimate the errors. The result is, by a calculation analogous to Eqs. (3.41)–(3.46) in [15],

$$\begin{aligned}\langle\Psi|H|\Psi\rangle - \frac{ne^\perp}{h^2} \ &\geq \ \int \sum_{i=1}^n \Big[\varepsilon|\nabla_i^\parallel F|^2 + (1-\varepsilon)|\nabla_i^\parallel F|^2 \chi_{\min_{k\neq i} |x_i - x_k| \geq R}(x_i) \\ &\qquad + \tfrac{1}{2}\sum_{j\neq i} W(x_i - x_j)|F|^2\Big] \prod_{k=1}^n dx_k \,,\end{aligned} \tag{5.8}$$

where W is obtained by averaging $a'U_R$ over z:

$$W(x - x') = 2a'' \int_{\mathbb{R}\times\mathbb{R}} s_h(z)^2 s_h(z')^2 U_R(|\mathbf{x} - \mathbf{x}'|)\chi_{\mathcal{B}_\delta}(z'/h) dz dz' \ . \tag{5.9}$$

Here, $a'' = a'(1-\eta)$ with an error term η containing the error estimates from the Temple inequality and from ignoring other points than the nearest neighbor to $\mathbf{x}_i$. Moreover, since $\int U_R(\mathbf{x})d\mathbf{x} = 4\pi$, $U_R(|\mathbf{x} - \mathbf{x}'|) = 0$ for $|\mathbf{x} - \mathbf{x}'| > R$, and $|s(z)^2 - s(z')^2| \leq R\|\partial_z s^2\|_\infty$ for $|z - z'| \leq R$ we have the estimate

$$\begin{aligned}\int_{\mathbb{R}^2} W(x)dx \ &\geq \ \frac{8\pi a''}{h}\left(\int_{\mathcal{B}_\delta} s(z)^4 dz - \frac{R}{h}\|\partial_z s^2\|_\infty\right) \\ &\geq \ \frac{8\pi a''}{h}\left(\|s\|_4^4 - \delta - \frac{R}{h}\|\partial_z s^2\|_\infty\right).\end{aligned} \tag{5.10}$$

As we will explain in a moment, the errors, and the replacement of $n-1$ by n, require the following terms to be small:

$$\frac{nh^2a}{\varepsilon R^3}, \quad \frac{nR}{h}, \quad \varepsilon, \quad \frac{1}{n}, \quad \delta, \quad \frac{R}{h\delta}. \tag{5.11}$$

The rationale behind the first term is as follows. The Temple errors in the averaging procedure at fixed $\mathbf{x}_1, \ldots, x_n$ produces a factor similar to (2.9), namely

$$\left(1 - a' \frac{\langle U^2 \rangle}{\langle U \rangle} \frac{1}{(\text{const.})\varepsilon/h^2 - (\text{const.})a'\langle U \rangle}\right) \tag{5.12}$$

with

$$\langle U^m \rangle = \int \left(\sum_{i=1}^{n} U(|\mathbf{x}_i - \mathbf{x}_{k(i)}|) \chi_{\mathcal{B}_\delta}(\mathbf{x}^{\perp}_{k(i)}/r) \right)^m \prod_{j=1}^{n} s_h(z_j)^2 dz_j, \tag{5.13}$$

cf. Eqs. (3.40)–(3.41) in [15]. The analogue of Eq. (3.42) in [15] is

$$\langle U \rangle \leq (\text{const.}) n(n-1) \frac{\|s\|_4^4}{hR^2}, \tag{5.14}$$

and $\langle U^2 \rangle \leq (\text{const.}) nR^{-3} \langle U \rangle$ by Schwarz's inequality. Since the denominator in (5.12) must be positive, we see in particular that the particle number must satisfy

$$n(n-1) < (\text{const.}) \frac{\varepsilon R^2}{ah} \tag{5.15}$$

and the error is of the order $nh^2a/\varepsilon R^3$ as claimed in (5.11).

The estimate from below on $\langle U \rangle$, obtained in the same way as Eq. (3.46) in [15], is

$$\begin{aligned} \langle U \rangle \;&\geq\; \sum_{i \neq j} \int U(|\mathbf{x}_i - \mathbf{x}_j|) \chi_{\mathcal{B}_\delta}(z_j/h) \left(1 - \sum_{k,k \neq i,j} \theta(R - |\mathbf{x}_k - \mathbf{x}_i|)\right) \prod_{l=1}^{n} s_h(z_l)^2 dz_l \\ &\geq\; \frac{1}{2a''} \sum_{i \neq j} W(x_i - x_j) \left(1 - (n-2)\frac{R}{h}\|s\|_\infty^2\right) . \end{aligned} \tag{5.16}$$

In particular, the second term in (5.11), nR/h, should be small. The requirement that ε and n^{-1} are small needs no further comments.

The potential W can be written as

$$W(x) = \lambda R^{-2} w_1(x/R)\,, \tag{5.17}$$

where w_1 is independent of R, with

$$\int w_1(x) dx = 1 \tag{5.18}$$

and

$$\lambda = \frac{8\pi\ a \int s^4}{h}(1 - \eta')\,. \tag{5.19}$$

Here, η' is an error term involving δ and $R/(h\delta)$ (cf. (5.5) and (5.10)) besides the other terms in (5.11). In particular, δ and $R/(h\delta)$ should be small. The 2D scattering length of (5.17) can

be computed by Corollary 4.2 and has the right form (1.17) to leading order in λ. (Recall from the remark preceding Eq. (1.19) that R in (4.12) can be replaced by h as long as $ca < R < Ch$.)

The Hamiltonian on the right side of Eq. (5.8) can now be treated with the methods of [19]. The only difference from the Hamiltonian discussed in that paper is the reduced kinetic energy inside the range of the potential W. This implies that λ in the error term $\eta(\lambda)$ in Corr. 4.2 should be replaced by λ/ε, which requires

$$\frac{a}{\varepsilon h} \ll 1. \tag{5.20}$$

Otherwise the method is the same as in [19]: a slight modification of Dyson's Lemma (lemma 3.1 in [19]) allows to substitute for W a potential $\tilde{U}$ of larger range, $\tilde{R}$, to which perturbation theory and Temple's inequality can be applied. The modified Dyson lemma is discussed in the Appendix. The fraction of the kinetic energy borrowed for the application of Temple's inequality as in Eq. (3.16) in [19] will be denoted by $\tilde{\varepsilon}$. The errors that have now to be controlled are

$$\tilde{\varepsilon}, \quad n\tilde{R}^2/\ell^2, \quad \frac{R}{\tilde{R}}, \quad \frac{n\ell^2}{\tilde{\varepsilon}\tilde{R}^2 \ln(\tilde{R}^2/a_{2\mathrm{D}}^2)}. \tag{5.21}$$

To explain these terms we refer to Eqs. (3.18)–(3.19) in [19] which contain the relevant estimates. Substituting $\tilde{\varepsilon}$ for ε and $\tilde{R}$ for R in these inequalities we see first that $\tilde{\varepsilon}$ and $n\tilde{R}^2/\ell^2$ should be small. The smallness of $R/\tilde{R}$ guarantees that the "hole" of radius R in the 2D Dyson potential has negligible effect. Since the denominator in the Temple error in Eq. (3.19) in [19] must be positive we see also that the particle number n in the box should obey the bound

$$n(n-1) < (\text{const.})\, \tilde{\varepsilon} \ln(\tilde{R}^2/a_{2\mathrm{D}}^2), \tag{5.22}$$

and the Temple error is bounded by $(\text{const.})n\ell^2/(\tilde{\varepsilon}\tilde{R}^2 \ln(\tilde{R}^2/a_{2\mathrm{D}}^2))$.

We summarize the discussion so far in the following Lemma:

Lemma 5.1. *For all n satisfying* (5.15) *and* (5.22)

$$E_{\mathrm{box}}(n,\ell,h,a) \geq \frac{2\pi n(n-1)}{\ell^2} |\ln(a_{2\mathrm{D}}^2/\tilde{R}^2)|^{-1} \Big(1 - \mathcal{E}(n,\ell,h,a;\varepsilon,R,\tilde{\varepsilon},\tilde{R},\delta)\Big) \tag{5.23}$$

where $\mathcal{E}$ tends to zero together with terms listed in (5.11) *and* (5.21).

We note also from Eq. (3.19) in [19] that $K(n) = (1-\mathcal{E})$ is decreasing in n (for the other parameters fixed). If $K(n)$ is defined as zero for n not satisfying (5.15) and (5.22) the estimate (5.23) hold for all n.

5.2 Global bound, uniform case

Using superadditivity of the energy (which follows from $W \geq 0$), monotonicity of $K(n)$ and convexity of $n(n-1)$ in the same way as in Eqs. (7)–(12) in [18], one sees that if $\rho = N/L^2$ is the density in the thermodynamically large box of side length L the optimal choice of n in the box of fixed side length ℓ is $n \sim \rho\ell^2$. We thus have to show that it is possible to choose the parameters ε, R, δ, $\tilde{\varepsilon}$ and $\tilde{R}$ and ℓ,in such a way that all the error terms (5.11) and (5.21), as well as δ and $R/(h\delta)$ are small, while $|\ln(a_{2\mathrm{D}}^2/\tilde{R}^2)| = |\ln(a_{2\mathrm{D}}^2\rho)|(1+o(1))$. We note that the conditions $a/h \ll 1$ and $\rho|\ln(\rho a_{2\mathrm{D}}^2)|^{-1} \ll 1/h^2$ imply $\rho a_{2\mathrm{D}}^2 \to 0$ and hence $|\ln(\rho a_{2\mathrm{D}}^2)|^{-1} \to 0$.

We make the ansatz

$$\varepsilon = \left(\frac{a}{h}\right)^{\alpha}, \quad \delta = \left(\frac{a}{h}\right)^{\alpha'}, \quad R = h\left(\frac{a}{h}\right)^{\beta}, \tag{5.24}$$

and choose ℓ such that L is a multiple of ℓ with

$$\rho^{-1/2} \ll \ell \lesssim \rho^{-1/2} \left(\frac{a}{h}\right)^{-\gamma}. \tag{5.25}$$

Then $n = \rho\ell^2 \gg 1$ and $R/(h\delta) = (a/h)^{\beta-\alpha'}$. The error terms (5.11) are also powers of a/h and we have to ensure that all exponents are positive, in particular

$$\beta - \alpha' > 0, \quad \beta - 2\gamma > 0, \quad 1 - \alpha - 3\beta - 2\gamma > 0. \tag{5.26}$$

This is fulfilled, e.g., for

$$\alpha = \alpha' = \frac{1}{9}, \quad \beta = \frac{2}{9}, \quad \gamma = \frac{1}{18}, \tag{5.27}$$

with all the exponents (5.26) equal to 1/9. Note also that with this choice Eq. (5.20) is fulfilled.

Next we write

$$\tilde{\varepsilon} = |\ln(\rho a_{2\mathrm{D}}^2)|^{-\kappa}, \quad \tilde{R} = \rho^{-1/2} |\ln(\rho a_{2\mathrm{D}}^2)|^{-\zeta}. \tag{5.28}$$

Then $|\ln(a_{2\mathrm{D}}^2/\tilde{R}^2)| = |\ln(a_{2\mathrm{D}}^2\rho)|(1 + o(1))$. The error terms (5.21) are

$$\tilde{\varepsilon} = |\ln(\rho a_{2\mathrm{D}}^2)|^{-\kappa}, \quad \frac{R}{\tilde{R}} = \left(\frac{a}{h}\right)^{\beta} (\rho h^2)^{1/2} |\ln(\rho a_{2\mathrm{D}}^2)|^{\zeta}, \quad \frac{n\tilde{R}^2}{\ell^2} = |\ln(\rho a_{2\mathrm{D}}^2)|^{-2\zeta}, \tag{5.29}$$

and

$$\frac{n\ell^2}{\tilde{\varepsilon}\tilde{R}^2 \ln(\tilde{R}^2/a_{2\mathrm{D}}^2)} = \left(\frac{a}{h}\right)^{-4\gamma} |\ln(\rho a_{2\mathrm{D}}^2)|^{-(1-\kappa-2\zeta)} (1 + O(\ln|\ln(\rho a_{2\mathrm{D}}^2)|)). \tag{5.30}$$

Since $(a/h)^{-4\gamma} |\ln(\rho a_{2\mathrm{D}}^2)|^{-4\gamma} = O(1)$, the error term (5.30) can also be written as

$$\frac{n\ell^2}{\tilde{\varepsilon}\tilde{R}^2 \ln(\tilde{R}^2/a_{2\mathrm{D}}^2)} = O(1) |\ln(\rho a_{2\mathrm{D}}^2)|^{-(1-\kappa-2\zeta-4\gamma)} (1 + O(\ln|\ln(\rho a_{2\mathrm{D}}^2)|)). \tag{5.31}$$

The condition $\rho h^2 < C$ is used to bound $R/\tilde{R}$ in (5.29). Namely,

$$(\rho h^2)^{1/2} |\ln(\rho a_{2\mathrm{D}}^2)|^{\zeta} \leq (\text{const.})(h/a)^{\zeta}, \tag{5.32}$$

so

$$\frac{R}{\tilde{R}} = O(1) \left(\frac{a}{h}\right)^{\beta-\zeta}. \tag{5.33}$$

We choose now

$$\zeta = \frac{1}{9}, \quad \kappa = \frac{2}{9}. \tag{5.34}$$

Then

$$\beta - \zeta = \frac{1}{9} \quad \text{and} \quad 1 - \kappa - 2\zeta - 4\gamma = \frac{1}{3}. \tag{5.35}$$

This completes our discussion of the lower bound of the energy in the thermodynamic limit, i.e., in Theorem 1.4, for the case $\rho h^2 \leq C < \infty$. As already mentioned, the case $\rho h^2 \gg 1$ can be treated with the 3D methods of [18, 13], taking care to retain uniformity in the parameter h/ℓ. The analogous problem for the reduction form 3D to 1D is discussed in detail in [15], cf. in particular the lower bound in Theorem 3.2 in [15]. Since no new aspects arise in the reduction from 3D to 2D we refrain from discussing the case $\rho h^2 \gg 1$ further here.

5.3 Global bound in a trap and BEC

We now discuss the case when the system is inhomogeneous in the x variables, i.e., the lower bound for Theorem 1.1. A possible approach, analogous to that of [13], is to modify Eq. (5.2) and write a general wave function as

$$\Psi(\mathbf{x}_1,\dots,\mathbf{x}_n) = f(\mathbf{x}_1,\dots,\mathbf{x}_n)\prod_{k=1}^{n}\varphi^{\mathrm{GP}}(x_k)s_h(z_k)\ , \tag{5.36}$$

This is always possible because φ^{GP} and s_h are strictly positive. As in [13] the task then becomes to minimize a quadratic form in f involving the GP density $\rho^{\mathrm{GP}}(x_i) = \varphi^{\mathrm{GP}}(x_i)^2$ both as a weight factor in the integration over the x_i's and also as a replacement for the external potential $V_L(x_i)$. There is, however, a neat alternative approach to deal with inhomogeneities due to R. Seiringer [36] that is somewhat simpler and that we shall follow with appropriate modifications. This approach leads also quickly to a proof of Theorem 1.3.

5.3.1 GP case

We start by writing again the wave function in the form (5.2) but this time with $n = N$. With H including the trapping potential V_L, cf. (1.3), and using the symmetry of F and f we obtain as in Eq. (5.4)

$$\begin{aligned}\langle\Psi|H|\Psi\rangle - \frac{Ne^{\perp}}{h^2} \ &\geq T + N\int V_L(x_1)F^2\prod_{k=1}^{N}dx_k \\ &\quad + N\int\left[\varepsilon|\partial_1 f|^2 + a'U_R(|\mathbf{x}_1-\mathbf{x}_{k(1)}|)\chi_{\mathcal{B}_\delta}(z_{k(1)}/h)|f|^2\right]\prod_{k=1}^{N}s_h(z_k)^2 d\mathbf{x}_k,\end{aligned} \tag{5.37}$$

where

$$T = N\left[\varepsilon\int_{|x_1-x_{k(1)}|\leq R}|\nabla_1^{\parallel}\Psi|^2\prod_{k=1}^{N}d\mathbf{x}_k + \int_{|x_1-x_{k(1)}|\geq R}|\nabla_1^{\parallel}\Psi|^2\prod_{k=1}^{N}d\mathbf{x}_k\right]. \tag{5.38}$$

Note that we have here not yet used Eq. (5.6) to bound the parallel kinetic energy of f in terms of that of F. The reason is that we want to prove BEC for the wave function Ψ and not only for F.

The next step is to split the kinetic energy T, for given $\tilde{R} > R$ and $\tilde{\varepsilon} > 0$, into two parts with the main contributions coming from length scales $> \tilde{R}$ and $< \tilde{R}$ respectively in the longitudinal directions. To write this in a compact way we introduce the notation

$$X = (x_2,\dots,x_N),\quad dX = dx_2\cdots dx_N;\quad Z = (z_1,...,z_N),\quad dZ = dz_1\cdots dz_N \tag{5.39}$$

and

$$\begin{aligned}\Omega_X^{<R} = \{x_1 \,:\, |x_1-x_{k(1)}|\leq R\},\quad \Omega_X^{R,\tilde{R}} = \{x_1 \,:\, R\leq|x_1-x_{k(1)}|\leq\tilde{R}\},\\ \Omega_X^{>\tilde{R}} = \{x_1 \,:\, |x_1-x_{k(1)}|\geq\tilde{R}\}.\end{aligned} \tag{5.40}$$

The splitting of the kinetic energy is

$$T \geq T^{>} + T^{<} \tag{5.41}$$

with

$$T^{>} = N \int_{\mathbb{R}^{3N-2}} \left[\frac{\varepsilon\tilde{\varepsilon}}{2} \int_{\Omega_X^{<R}} |\nabla_1^{\parallel}\Psi|^2 + \frac{\tilde{\varepsilon}}{2} \int_{\Omega_X^{R,\tilde{R}}} |\nabla_1^{\parallel}\Psi|^2 + \big(1 - \frac{\tilde{\varepsilon}}{2}\big) \int_{\Omega_X^{>\tilde{R}}} |\nabla_1^{\parallel}\Psi|^2 \right] dX dZ, \tag{5.42}$$

$$T^{<} = N \int_{\mathbb{R}^{2(N-1)}} \left[\varepsilon\big(1 - \frac{\tilde{\varepsilon}}{2}\big) \int_{\Omega_X^{<R}} |\nabla_1^{\parallel}F|^2 + \big(1 - \frac{\tilde{\varepsilon}}{2}\big) \int_{\Omega_X^{R,\tilde{R}}} |\nabla_1^{\parallel}F|^2 + \int_{\Omega_X^{>\tilde{R}}} \frac{\tilde{\varepsilon}}{2} |\nabla_1^{\parallel}F|^2 \right] dX. \tag{5.43}$$

Here we have used (5.6) for $T^{<}$, but not for $T^{>}$. We then add and subtract $8\pi N^2 g \int_{\mathbb{R}^{2N}} \rho^{\mathrm{GP}}(x_1)|F|^2$ from the right side of (5.37) and, using (5.41), estimate it by $E^{>} + E^{<}$ with

$$E^{>} = T^{>} + N \int V_L(x_1) F^2 + 8\pi N^2 g \int \rho^{\mathrm{GP}}(x_1)|F|^2, \tag{5.44}$$

$$E^{<} = T^{<} + N \int \left[\varepsilon |\partial_1 f|^2 + a' U_R(|\mathbf{x}_1 - \mathbf{x}_{k(1)}|) \chi_{\mathcal{B}_\delta}(z_{k(1)}/h) |f|^2 \right] \prod_{k=1}^{N} s_h(z_k)^2 d\mathbf{x}_k \\ - 8\pi N^2 g \int \rho^{\mathrm{GP}}(x_1)|F|^2. \tag{5.45}$$

The two terms are now considered separately. The first remark is that $\varphi^{\mathrm{GP}}(x)$ is the ground state wave function of the 1-body Hamiltonian

$$-\Delta^{\parallel} + V_L(x) + 8\pi N g \rho^{\mathrm{GP}}(x) \tag{5.46}$$

and the energy is equal to the GP chemical potential

$$\mu^{\mathrm{GP}}(Ng) = E_{\mathrm{2D}}^{\mathrm{GP}}(1, L, Ng) + 4\pi N g \int \varphi^{\mathrm{GP}}(x)^4 dx, \tag{5.47}$$

cf. Section 2 in [14]. Hence, if $T^{>}$ were replaced by the full kinetic energy, $\int |\nabla_1^{\parallel}\Psi|^2$, then $E^{>}$ would be bounded below by $N\mu^{\mathrm{GP}}(Ng) = E_{\mathrm{2D}}^{\mathrm{GP}}(N, L, g) + 4\pi N^2 g \int \varphi^{\mathrm{GP}}(x)^4 dx$. Using a variant of the generalized Poincaré inequalities of [16] it is shown in [35, 36] that this estimate also holds for $E^{>}$ up to small errors. Even a stronger result is true (cf. [36], Eq. (51)), because a contribution form the one particle density matrix in the lontitudinal direction,

$$\gamma_{\Psi}^{\parallel}(x, x') = \int_{\mathbb{R}} \gamma_{\Psi}((x, z), (x', z)) dz \tag{5.48}$$

can be included: Provided $\tilde{R}^2 N \to 0$ for $N \to \infty$, and Ng is fixed,

$$E^{>}/N \geq \left(\mu^{\mathrm{GP}}(Ng) + C \mathrm{Tr} \left[\tfrac{1}{N} \gamma_{\Psi}^{\parallel} (1 - |\varphi^{\mathrm{GP}}\rangle\langle\varphi^{\mathrm{GP}}|) \right] \right) (1 - o(1)) \tag{5.49}$$

where the term $o(1)$ goes to zero if $\varepsilon, \tilde{\varepsilon} \to 0$ go to zero after the limit $N \to \infty$ has been taken.

The term $E^{<}$ is treated in the same manner as in the homogeneous case by introducing boxes, Λ_α, of side length ℓ in the longitudinal directions. Defining

$$\rho_\alpha = \sup_{x \in \Lambda_\alpha} \rho^{\mathrm{GP}}(x) \tag{5.50}$$

we have (cf. Eq. (60) in [36])

$$E^{<} \geq \inf_{n_\alpha} \sum_{\alpha} (E(n_\alpha, \ell) - 8\pi N g \rho_\alpha n_\alpha) \tag{5.51}$$

with $E(n,\ell)$ the right side of (5.23). (It depends also on the parameters $h, a; \epsilon, R, \varepsilon, \tilde{R}, \delta$.) The infimum is taken over all distributions of the N particles among the boxes and n_α denotes the particle number in box α. Choosing the box length as in Eq. (5.25) with ρ replaced by the mean density $\bar{\rho} \sim N/\bar{L}$, and arguing exactly as in [36], Eqs. (71)–(78) we obtain

$$E^< \geq -4\pi Ng \int |\varphi^{\mathrm{GP}}(x)|^4 dx (1 + o(1)). \tag{5.52}$$

Adding (5.52) and (5.49) we obtain the bound

$$\frac{\langle \Psi | H | \Psi \rangle}{N} - \frac{e^\perp}{h^2} \geq \left[E^{\mathrm{GP}}_{\mathrm{2D}}(1, L, Ng) + C\mathrm{Tr}[\tfrac{1}{N}\gamma^{\|}_\Psi (1 - |\varphi^{\mathrm{GP}}\rangle\langle\varphi^{\mathrm{GP}}|) \right] (1 - o(1)). \tag{5.53}$$

If Ψ is the ground state wave function Ψ_0, then (5.53) together with the upper bound (3.61) proves both the convergence of the energy as claimed in Theorem 1.1 and also the convergence of the density matrix $\gamma^{\|}_\Psi(x,x')$ to $\varphi^{\mathrm{GP}}(x)\varphi^{\mathrm{GP}}(x')$ (in trace norm) as $N \to \infty$ for the case that Ng is fixed (or at least bounded). The convergence of the full density matrix $\gamma_{\Psi_0}(\mathbf{x}, \mathbf{x}')$, scaled by h in the variables z and z', to $\varphi^{\mathrm{GP}}(x)\varphi^{\mathrm{GP}}(x')s(z)s(z')$ follows from a simple energetic consideration: The energy gap between the lowest and the first excited level of $-\partial_z^2 + V_h^\perp$ is large compared to $\langle \Psi | H | \Psi \rangle / N - \frac{e^\perp}{h^2}$ in the limit considered. Hence the contributions to the density matrix form the excited states in the perpendicular direction must vanish.

Remark. The above proof of BEC can be extended to allow a slow increase of Ng to infinity with N, but this improvement is restricted by the available estimates of the error terms in (5.49) and (5.53) and is only marginal.

5.3.2 TF case

To complete the proof of Theorem 1.1 one must also consider the case that $Ng \to \infty$, i.e. the 'Thomas-Fermi' limit. Here the starting point is again (5.37), this time with T replaced by the smaller quantity

$$T' = \left[\varepsilon \int_{|x_1 - x_{k(1)}| \leq R} |\nabla_i^{\|} F|^2 \prod_{k=1}^N dx_k + \int_{|x_1 - x_{k(1)}| \geq R} |\nabla_i^{\|} F|^2 \prod_{k=1}^N dx_k \right] \tag{5.54}$$

cf. (5.6). We then use the TF equation

$$\rho^{\mathrm{TF}}(x) = \frac{1}{8\pi Ng} [\mu^{\mathrm{TF}} - V_L(x)]_+ \tag{5.55}$$

to trade the potential $V_L(x)$ for the TF density $\rho^{\mathrm{TF}}(x)$ which is the square of the minimizer of (1.10) with the kinetic term omitted. The TF chemical potential μ^{TF} is determined by the normalization condition $\int \rho^{\mathrm{TF}} = 1$ and satisfies in analogy to (5.47)

$$\mu^{\mathrm{TF}}(Ng) = E^{\mathrm{TF}}_{\mathrm{2D}}(1, L, Ng) + 4\pi Ng \int \varphi^{\mathrm{TF}}(x)^4 dx, \tag{5.56}$$

From (5.55) it follows that

$$V_L(x) \geq \mu^{\mathrm{TF}} - 8\pi(Ng)\rho^{\mathrm{TF}}(x). \tag{5.57}$$

Both μ^{TF} and ρ^{TF} depend on Ng and scale in a simple way if the potential $V_L(x)$ is a homogeneous function of some degree $p > 0$, cf. Eqs. (2.20)-(2.21) in [15]. We can now apply the box

method to the minimization of

$$T' + \mu^{\mathrm{TF}} - 8\pi(Ng)N \int \rho^{\mathrm{TF}}(x_1) F^2 \prod_{k=1}^{N} dx_k$$
$$+ N \int \left[\varepsilon |\partial_1 f|^2 + a' U_R(|\mathbf{x}_1 - \mathbf{x}_{k(1)}|) \chi_{\mathcal{B}_\delta}(z_{k(1)}/h) |f|^2\right] \prod_{k=1}^{N} s_h(z_k)^2 d\mathbf{x}_k, \quad (5.58)$$

obtaining in analogy to (5.51) and using (5.56),

$$(5.58) \geq E^{\mathrm{TF}}_{\mathrm{2D}}(N, L, g) + 4\pi(Ng)N \int \rho^{\mathrm{TF}}(x)^2 dx + \inf_{n_\alpha} \sum_\alpha (E(n_\alpha, \ell) - 8\pi N g \rho^{\mathrm{TF}}_\alpha n_\alpha). \quad (5.59)$$

The remaining steps towards the estimate

$$(5.58) \geq E^{\mathrm{TF}}_{\mathrm{2D}}(N, L, g)(1 - o(1)) \quad (5.60)$$

are now exactly like in Section 4 in [14].

6 Conclusions

We have investigated the dimensional reduction of a trapped, interacting Bose gas when the trap potential is strongly confining in one direction. Starting from the many-body Hamiltonian with repulsive, short range interactions we have shown rigorously how an effective 2D description of the ground state energy and density emerges and how the parameters of the 2D gas relate to those of the 3D gas. Two parameter regimes can be distinguished: One where the gas retains some of its 3D character despite the tight confinement and another where the situation is manifestly two-dimensional with a logarithmic dependence of the coupling parameter on the density. Moreover, we have shown that the trapped gas is Bose-Einstein condensed in the ground state provided the coupling parameter Ng stays bounded.

A Appendix: A modification of Dyson's lemma

Let $W(x) \geq 0$ be a rotationally symmetric potential in 2D with $W(x) = 0$ for $|x| > R$. Denote by B_R the ball of radius R around the origin. For $0 < \varepsilon$ and $R' \geq R$ define

$$E_{R',\varepsilon} = \min \int_{B_R} \left[\varepsilon |\nabla \phi(x)|^2 + \tfrac{1}{2} W(x) |\phi(x)|^2\right] dx + \int_{B_{R'} \setminus B_R} |\nabla \phi(x)|^2 dx \quad (A.1)$$

where the minimum is taken over $\phi \in H^1(B_{R'})$ with $\phi(x) = 1$ for $|x| = R'$.

Lemma A.1.

$$E_{R',\varepsilon} = \frac{2\pi}{\ln(R'/R) + 2\pi/E_{R,\varepsilon}} \quad (A.2)$$

Proof. For any $c > 0$ the minimum of the first integral in (A.1) with boundary condition $\phi(x) = c$ for $|x| = R$ is $c^2 E_{R,\varepsilon}$ and the minimum of the second integral with the same boundary condition at $|x| = R$ and $\phi(x) = 1$ for $|x| = R'$ is $2\pi \ln(R'/R)/(\ln(\tilde{R}/b))^2$ where b is determined by $\ln(R/b)/\ln(R'/b) = c$. Adding the two contributions and minimizing over c gives (A.2). □

Lemma A.2 (Modified Dyson Lemma). *Let W be as above, let $\tilde{R} > R$ and let $\tilde{U}(r) \geq 0$ be any function with support in $[R, \tilde{R}]$ satisfying*

$$2\pi \int_R^{\tilde{R}} \tilde{U}(R') E_{R',\varepsilon}^{-1} R' dR' \leq 1 \tag{A.3}$$

with $E_{R',\varepsilon}$ as in (A.2)*. Let $\mathcal{B} \subset \mathbb{R}^2$ be star-shaped with respect to 0. Then, for all functions $\phi \in H^1(\mathcal{B})$,*

$$\int_{\mathcal{B}\cap B_R} \left[\varepsilon|\nabla\phi(x)|^2 + \frac{1}{2}W(x)|\phi(x)|^2\right] dx + \int_{\mathcal{B}\cap(B_{\tilde{R}}\setminus B_R)} |\nabla\phi(x)|^2 dx \\ \geq \int_{\mathcal{B}} \tilde{U}(|x|)|\phi(x)|^2 \, dx. \tag{A.4}$$

Proof. The proof is very similar to that of Lemma 3.1 in [19]. In polar coordinates, r, θ, one has $|\nabla\phi|^2 \geq |\partial\phi/\partial r|^2$ so it suffices to prove the analogue of (A.4) for each angle $\theta \in [0, 2\pi)$. Denote $\phi(r, \theta)$ simply by $f(r)$, and let $R(\theta)$ denote the distance of the origin to the boundary of $\mathcal{B}$ along the ray θ. It suffices to consider the case $R \leq R(\theta)$ (here, $W \geq 0$ is used) and the estimate to prove is

$$\int_0^R \left\{\varepsilon|\partial f(r)/\partial r|^2 + \tfrac{1}{2}W(r)|f(r)|^2\right\} \, r dr + \int_R^{\min\{\tilde{R},R(\theta)\}} |\partial f(r)/\partial r|^2 r \, dr \\ \geq \int_R^{\min\{\tilde{R},R(\theta)\}} \tilde{U}(r)|f(r)|^2 \, r dr. \tag{A.5}$$

For the given value of θ, consider the disc $B_{R(\theta)}$ centered at the origin in $\mathbb{R}^2$ and of radius $R(\theta)$. Our function f defines a rotationally symmetric function, $x \mapsto f(|x|)$ on $B_{R(\theta)}$, and (A.5) is equivalent to

$$\int_{B_{R(\theta)}\cap B_{R_0}} \left[\varepsilon|\nabla f(|x|)|^2 + \frac{1}{2}W(r)|f(|x|)|^2\right] dx + \int_{B_{R(\theta)}\cap(B_{\tilde{R}}\setminus B_{R_0})} |\nabla f(|x|)|^2 dx \\ \geq \int_{B_{R(\theta)}} \tilde{U}(|x|)|f(|x|)|^2 dx \tag{A.6}$$

If $R \leq R' \leq \min\{\tilde{R}, R(\theta)\}$ the left side of (A.6) is not smaller than the same quantity with $B_{R(\theta)}$ replaced by the smaller disc $B_{R'}$. (Again, $W \geq 0$ is used.) According to (A.1) this integral over $B_{R'}$ is at least $E_{R',\varepsilon}|f(R')|^2$. Hence, for every such R',

$$\int_0^R \left\{\varepsilon|\partial f(r)/\partial r|^2 + \tfrac{1}{2}W(r)|f(r)|^2\right\} \, r dr + \int_R^{\min\{\tilde{R},R(\theta)\}} |\partial f(r)/\partial r|^2 r \, dr \\ \geq (2\pi)^{-1} E_{R',\varepsilon}|f(R')|^2 \tag{A.7}$$

The proof is completed by multiplying both sides of (A.7) by $2\pi\tilde{U}(R')E_{R',\varepsilon}^{-1}R'$ and, finally, integrating with respect to R' from R to $\min\{\tilde{R}, R(\theta)\}$. □

A convenient choice for the Dyson potential $\tilde{U}$ is

$$\tilde{U}(r) = \begin{cases} \nu(\tilde{R})^{-1} & \text{if } R \leq r \leq \tilde{R} \\ 0 & \text{otherwise ,} \end{cases} \tag{A.8}$$

with

$$\nu(\tilde{R}) = 2\pi \int_R^{\tilde{R}} E_{R',\varepsilon} R' \, dR'. \tag{A.9}$$

In the case considered in this paper the potential W is integrable with coupling constant $\lambda \sim a/h(1 + o(1))$ while $\varepsilon \sim (a/h)^{1/9}$. In particular $\lambda/\varepsilon \to 0$ as $a/h \to 0$. From Section 4 and the discussion of Eqs. (5.19)–(5.20) we thus conclude that in our case

$$E_{R,\varepsilon} = 2\pi / \ln(R/a_{2\mathrm{D}})(1 + o(1)). \tag{A.10}$$

By Eq. (A.2) we then also have

$$E_{R',\varepsilon} = 2\pi / \ln(R'/a_{2\mathrm{D}})(1 + o(1)). \tag{A.11}$$

and thus

$$\nu(\tilde{R}) = \tfrac{1}{4}\tilde{R}^2 \ln(\tilde{R}^2/a_{2\mathrm{D}}^2)(1 + o(1)). \tag{A.12}$$

Thus, for R and ε chosen as in (5.24) and (5.27), the modified Dyson Lemma gives, up to negligible errors, exactly the same result as the standard 2D Dyson Lemma for a potential with scattering length $a_{2\mathrm{D}}$.

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